

# Cyclic, Simple, and Indecomposable Three-fold Triple Systems

Nabil Shalaby

Bradley Sheppard

Daniela Silvesan

Department of Mathematics and Statistics  
Memorial University of Newfoundland  
St. John's, Newfoundland  
CANADA A1C 5S7

April 3, 2014

**This paper is dedicated to the memory of Dr. Rolf Rees (1960-2013).**

## Abstract

In 2000, Rees and Shalaby constructed simple indecomposable two-fold cyclic triple systems for all  $v \equiv 0, 1, 3, 4, 7, \text{ and } 9 \pmod{12}$  where  $v = 4$  or  $v \geq 12$ , using Skolem-type sequences.

We construct, using Skolem-type sequences, three-fold triple systems having the properties of being cyclic, simple, and indecomposable for all admissible orders  $v$ , with some possible exceptions for  $v = 9$  and  $v = 24c + 57$ , where  $c \geq 2$  is a constant. To prove the simplicity we used a Mathematica computer program. We list in the Appendix the code and the results of the program.

**Keywords:** cyclic triple systems; Skolem-type sequences;  $\lambda$ -fold triple systems; indecomposable and simple designs

## 1 Introduction

A  $\lambda$ -fold triple system of order  $v$ , denoted by  $\text{TS}_\lambda(v)$ , is a pair  $(V, \mathcal{B})$  where  $V$  is a  $v$ -set of points and  $\mathcal{B}$  is a set of 3-subsets (*blocks*) such that any 2-subset of  $V$  appears in exactly  $\lambda$  blocks. An *automorphism group* of an  $\text{TS}_\lambda(v)$  is a permutation on  $V$  leaving  $\mathcal{B}$  invariant. An

$\text{TS}_\lambda(v)$  is *cyclic* if its automorphism group contains a  $v$ -cycle. If  $\lambda = 1$ , an  $\text{TS}_\lambda(v)$  is called *Steiner triple system* and is denoted by  $\text{STS}(v)$ . A cyclic  $\text{STS}(v)$  is denoted by  $\text{CSTS}(v)$ .

An  $\text{TS}_\lambda(v)$  is *simple* if it contains no repeated blocks. An  $\text{TS}_\lambda(v)$  is called *indecomposable* if its blocks set  $\mathcal{B}$  cannot be partitioned into sets  $\mathcal{B}_1, \mathcal{B}_2$  of blocks of the form  $\text{TS}_{\lambda_1}(v)$  and  $\text{TS}_{\lambda_2}(v)$ , where  $\lambda_1 + \lambda_2 = \lambda$  with  $\lambda_1, \lambda_2 \geq 1$ . A cyclic  $\text{TS}_\lambda(v)$  is called *cyclically indecomposable* if its block set  $\mathcal{B}$  cannot be partitioned into sets  $\mathcal{B}_1, \mathcal{B}_2$  of blocks to form a cyclic  $\text{TS}_{\lambda_1}(v)$  and  $\text{TS}_{\lambda_2}(v)$ , where  $\lambda_1 + \lambda_2 = \lambda$  with  $\lambda_1, \lambda_2 \geq 1$ .

The constructions of triple systems with the properties cyclic, simple, and indecomposable, were studied by many researchers for one property at a time; for example, cyclic triple systems for all  $\lambda$ s were constructed in [4, 12], simple for  $\lambda = 2$  in [15] and simple for every  $v$  and  $\lambda$  satisfying the necessary conditions in [6]. Also, some of the properties were combined in studies. For example, in [16], cyclic and simple two-fold triple systems for all admissible orders were constructed, while in [1, 5, 7, 9, 10, 18], simple and indecomposable designs for  $\lambda = 2, 3, 4, 5, 6$  and all admissible  $v$  were constructed. In [17], simple and indecomposable designs were constructed for all  $v \geq 24\lambda - 5$  satisfying the necessary conditions. For the general case of  $\lambda > 6$ , Colbourn and Colbourn [3] constructed a single indecomposable  $\text{TS}_\lambda(v)$  for each odd  $\lambda$ . Shen [14] used Colbourn and Colbourn result and some recursive constructions to prove the necessary conditions are asymptotically sufficient. Specifically, if  $\lambda$  is odd, then there exists a constant  $v_0$  depending on  $\lambda$  with an indecomposable simple  $\text{TS}_\lambda(v)$  design for all  $v \geq v_0$  satisfying the necessary conditions. In [8], the authors constructed two-fold cyclically indecomposable triple systems for all admissible orders. The authors also checked exhaustively the cyclic triple systems  $\text{TS}_\lambda(v)$  for  $\lambda = 2, v \leq 33$  and  $\lambda = 3, v \leq 21$  that are cyclically indecomposable and determined if they are decomposable (to non cyclic) or not.

In 2000, Rees and Shalaby [11] constructed simple indecomposable two-fold cyclic triple systems for all  $v \equiv 0, 1, 3, 4, 7, \text{ and } 9 \pmod{12}$  where  $v = 4$  or  $v \geq 12$  using Skolem-type sequences. They acknowledged that the analogous problem for  $\lambda > 2$  is more difficult.

In 1974, Kramer [9] constructed indecomposable three-fold triple systems for all admissible orders. We noticed that Kramer's construction for  $v \equiv 1 \text{ or } 5 \pmod{6}$  gives also cyclic and simple designs.

In this paper, we construct three-fold triple systems having the properties of being cyclic, simple, and indecomposable for all admissible orders  $v \equiv 3 \pmod{6}$ , except for  $v = 9$  and  $v = 24c + 57, c \geq 2$ .

## 2 Preliminaries

Let  $D$  be a multi set of positive integers with  $|D| = n$ . A *Skolem-type sequence of order  $n$*  is a sequence  $(s_1, \dots, s_t), t \geq 2n$  of  $i \in D$  such that for each  $i \in D$  there is exactly one  $j \in \{1, \dots, t-i\}$  such that  $s_j = s_{j+i} = i$ . Positions in the sequence not occupied by integers  $i \in D$  contain null elements. The null elements in the sequence are also called *hooks*, *zeros* or *holes*. As examples,  $(1, 1, 6, 2, 5, 2, 1, 1, 6, 5)$  is a Skolem-type sequence of order 5 and  $(7, 5, 2, 0, 2, 0, 5, 7, 1, 1)$  is a Skolem-type sequence of order 4.

Some special Skolem-type sequences are described below.

A *Skolem sequence of order  $n$*  is a sequence  $S_n = (s_1, s_2, \dots, s_{2n})$  of  $2n$  integers which satisfies the conditions:

1. for every  $k \in \{1, 2, \dots, n\}$  there are exactly two elements  $s_i, s_j \in S$  such that  $s_i = s_j = k$ , and
2. if  $s_i = s_j = k$ ,  $i < j$ , then  $j - i = k$ .

Skolem sequences are also written as collections of ordered pairs  $\{(a_i, b_i) : 1 \leq i \leq n, b_i - a_i = i\}$  with  $\cup_{i=1}^n \{a_i, b_i\} = \{1, 2, \dots, 2n\}$ .

For example,  $S_5 = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$  is a Skolem sequence of order 5 or, equivalently, the collection  $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$ .

Equivalently, a *Skolem sequence of order  $n$*  is a Skolem-type sequence with  $t = 2n$  and  $D = \{1, \dots, n\}$ .

A *hooked Skolem sequence of order  $n$*  is a sequence  $hS_n = (s_1, \dots, s_{2n-1}, s_{2n+1})$  of  $2n + 1$  integers which satisfies the above definition, as well as  $s_{2n} = 0$ .

As an example,  $hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, 0, 6)$  is a hooked Skolem sequence of order 6 or, equivalently, the collection  $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$ .

A *(hooked) Langford sequence of length  $n$  and defect  $d$* ,  $n > d$  is a sequence  $L_d^n = (l_i)$  of  $2n(2n + 1)$  integers which satisfies:

1. for every  $k \in \{d, d + 1, \dots, d + n - 1\}$ , there exist exactly two elements  $l_i, l_j \in L$  such that  $l_i = l_j = k$ ,
2. if  $l_i = l_j = k$  with  $i < j$ , then  $j - i = k$ ,
3. in a hooked sequence  $l_{2n} = 0$ .

We noticed that Kramer's construction [9] can be obtained using the canonical starter  $v - 2, v - 4, \dots, 3, 1, 1, 3, \dots, v - 4, v - 2$  and taking the base blocks  $\{0, i, b_i\}(\text{mod } v) | i = 1, 2, \dots, \frac{1}{2}(v - 1)$ . So, Kramer's construction can be obtained using Skolem-type sequences.

We prove next, that Kramer's construction for indecomposable triple systems produces simple designs.

**Theorem 2.1** [9] *The blocks  $\{0, \alpha, -\alpha\}(\text{mod } v) | \alpha = 1, \dots, \frac{1}{2}(v - 1)\}$  for  $v \equiv 1$  or  $5 \pmod{6}$  form a cyclic, simple, and indecomposable three-fold triple system of order  $v$ .*

**Proof** Let  $v = 6n + 1$ . The design is cyclic and indecomposable [9]. We prove that the cyclic three-fold triple systems produced by  $\{0, \alpha, -\alpha\}(\text{mod } v) | \alpha = 1, \dots, \frac{1}{2}(v - 1)\}$  is also simple.

Suppose that the construction above produces  $\{x, y, z\}$  as a repeated block. Any block  $\{x, y, z\}$  is of the form  $\{0, i, 6n + 1 - i\} + k$  for some  $i = 1, 2, \dots, \frac{1}{2}(v - 1)$  and  $k \in \mathbb{Z}_{6n+1}$ . Hence, if  $\{x, y, z\}$  is a repeated block we have

$$\{0, i_1, 6n + 1 - i_1\} + k_1 = \{0, i_2, 6n + 1 - i_2\} + k_2$$

whence,

$$\{0, i_2, 6n+1-i_2\} = \{0, i_1, 6n+1-i_1\} + k$$

for some  $i_1, i_2 \in \{1, 2, \dots, \frac{1}{2}(v-1)\}$  and some  $k \in \mathbb{Z}_{6n+1}$ .

If  $k = 0$ , we have  $i_2 = 6n+1-i_1$  and  $i_1 = 6n+1-i_2$ , which is impossible since  $6n+1-i_1 > i_2$  and  $6n+1-i_2 > i_1$  by definition (i.e.,  $i_1, i_2 \in \{1, 2, \dots, 3n\}$  while  $6n+1-i_1, 6n+1-i_2 \in \{3n+1, \dots, 6n\}$ ).

$$\text{If } k = i_2, \text{ we have } \begin{cases} i_1 + i_2 = 6n+1 \\ 6n+1-i_1+i_2 = 6n+1-i_2 \end{cases} \quad \text{or}$$

$$\begin{cases} i_1 + i_2 = 6n+1-i_2 \\ 6n+1-i_1+i_2 = 6n+1. \end{cases}$$

Since both  $i_1$  and  $i_2$  are at most  $3n$ , it is impossible to have  $i_1+i_2 = 6n+1$ . Also  $i_1 \neq i_2$ .

$$\text{If } k = 6n+1-i_2 \text{ we have } \begin{cases} i_1 + 6n+1-i_2 = 6n+1 \\ 6n+1-i_1+6n+1-i_2 = i_2+6n+1 \end{cases} \Leftrightarrow$$

$$\begin{cases} i_1 + i_2 = 6n+1-i_1 \\ 6n+1-i_2+i_1 = 6n+1 \end{cases} \quad \text{or} \quad \begin{cases} i_1 + 6n+1-i_2 = i_2 \\ 6n+1-i_1+6n+1-i_2 = 6n+1. \end{cases}$$

Since  $6n+1-i_2 > i_2$ , it is impossible to have  $i_1+6n+1-i_2 = i_2$ .

It follows that our design is simple. The case for  $v = 6n+5$  is similar. ■

In order to completely solve the case  $\lambda = 3$ , we have new constructions that give cyclic, simple, and indecomposable three-fold triple systems for  $v \equiv 3 \pmod{6}$ ,  $v \neq 9$  and  $v \neq 24c+57$ ,  $c \geq 2$ .

### 3 Simple Three-fold Cyclic Triple Systems

**Lemma 3.1** *For every  $n \equiv 0$  or  $1 \pmod{4}$ ,  $n \geq 8$ , there is a Skolem sequence of order  $n$  in which  $s_1 = s_2 = 1$  and  $s_{2n-2} = s_{2n} = 2$ .*

**Proof** To get a Skolem sequence of order  $n$  for  $n \equiv 0$  or  $1 \pmod{4}$ ,  $n \geq 8$ , take  $(1, 1, hL_3^{n-2})$ , replace the hook with a 2 and add the other 2 at the end of the sequence.

For  $n = 8$ , take  $hL_3^6 = (8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 0, 6)$ , for  $n = 12$  take  $hL_3^{10} = (9, 11, 3, 12, 4, 3, 7, 10, 4, 9, 8, 5, 11, 7, 6, 12, 5, 10, 8, 0, 6)$  and for the remaining  $hL_3^{n-2}$  hook a  $hL_4^{n-3}$  (see [13], Theorem 2, Case 1) to  $(3, 0, 0, 3)$ .

For  $n \equiv 1 \pmod{4}$ ,  $n \geq 9$ , take  $hL_3^{n-2}$  (see [13], Theorem 2, Case 1). ■

**Example 3.1** *From the above lemma we have  $S_8 = (1, 1, 8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 2, 6, 2)$ ,  $S_{12} = (1, 1, 9, 11, 3, 12, 4, 3, 7, 10, 4, 9, 8, 5, 11, 7, 6, 12, 5, 10, 8, 2, 6, 2)$  and  $S_{16} = (1, 1, 9, 6, 4, 14, 15, 11, 4, 6, 13, 9, 16, 7, 12, 10, 8, 5, 11, 14, 7, 15, 5, 13, 8, 10, 12, 3, 16, 2, 3, 2)$ .*

We use the following construction to get cyclic  $\text{TS}_3(2n+1)$  for  $n \equiv 0$  or  $1 \pmod{4}$ :

**Construction 3.1** [12] Let  $S_n = (s_1, s_2, \dots, s_{2n})$  be a Skolem sequence of order  $n$  and let  $\{(a_i, b_i) | 1 \leq i \leq n\}$  be the pairs of positions in  $S_n$  for which  $b_i - a_i = i$ . Then the set  $\mathcal{F} = \{\{0, i, b_i\} | 1 \leq i \leq n\} \pmod{2n+1}$  is a  $(2n+1, 3, 3) - DF$ . Hence, the set of triples in  $\mathcal{F}$  form the base blocks of a cyclic  $TS_3(2n+1)$ .

Then, we apply Construction 3.1 to the Skolem sequences given by Lemma 3.1 to get cyclic three-fold triple systems that are simple and indecomposable.

**Construction 3.2** Let  $S_n = (s_1, s_2, \dots, s_{2n})$  be a Skolem sequence of order  $n$  given by Lemma 3.1, and let  $\{(a_i, b_i) | 1 \leq i \leq n\}$  be the pairs of positions in  $S_n$  for which  $b_i - a_i = i$ . Then the set  $\mathcal{F} = \{\{0, i, b_i\} | 1 \leq i \leq n\} \pmod{2n+1}$  form the base blocks of a cyclic, simple, and indecomposable  $TS_3(2n+1)$ .

**Example 3.2** If we apply Construction 3.2 to the Skolem sequence of order 8:  $(1, 1, 8, 3, 5, 7, 3, 4, 6, 5, 8, 4, 7, 2, 6, 2)$  we get the base blocks  $\{\{0, 1, 2\}, \{0, 2, 16\}, \{0, 3, 7\}, \{0, 4, 12\}, \{0, 5, 10\}, \{0, 6, 15\}, \{0, 7, 13\}, \{0, 8, 11\}\} \pmod{17}$ . These base blocks form a cyclic  $TS_3(17)$  by Construction 3.1. We are going to prove next that this design is also indecomposable and simple.

**Theorem 3.2** The  $TS_3(6n+3)$ ,  $n \geq 2$  produced by applying Construction 3.2 are simple, except for  $v = 24c + 57$ ,  $c \geq 2$ .

**Proof** Let  $v = 2n+1$ ,  $n \equiv 0$  or  $1 \pmod{4}$ ,  $n \geq 8$ .

Suppose that the construction above produces  $\{x, y, z\}$  as a repeated block. With regards to Construction 3.2, any block  $\{x, y, z\}$  is of the form  $\{0, i, b_i\} + k$  for some  $i = 1, 2, \dots, n$  and  $k \in \mathbb{Z}_{2n+1}$ . Hence, if  $\{x, y, z\}$  is a repeated block we have

$$\{0, i_1, b_{i_1}\} + k_1 = \{0, i_2, b_{i_2}\} + k_2$$

whence,

$$\{0, i_2, b_{i_2}\} = \{0, i_1, b_{i_1}\} + k$$

for some  $i_1, i_2 \in \{1, 2, \dots, n\}$  and some  $k \in \mathbb{Z}_{2n+1}$ .

If  $k = 0$ , we have  $i_2 = b_{i_1}$  and  $i_1 = b_{i_2}$  which is impossible since  $b_{i_1} \geq i_1 + 1$  and  $b_{i_2} \geq i_2 + 1$  from the definition of a Skolem sequence.

$$\text{If } k = i_2, \text{ we have } \begin{cases} i_1 + i_2 = 2n+1 \\ b_{i_1} + i_2 = b_{i_2} \end{cases} \quad \text{or} \quad \begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n+1 \end{cases}.$$

Since both  $i_1$  and  $i_2$  are at most  $n$ , it is impossible to have  $i_1 + i_2 = 2n+1$ .

$$\text{If } k = b_{i_2}, \text{ we have } \begin{cases} i_1 + b_{i_2} = 2n+1 \\ b_{i_1} + b_{i_2} = i_2 + 2n+1 \end{cases} \Leftrightarrow \begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n+1 \end{cases} \quad \text{or} \quad \begin{cases} i_1 + b_{i_2} = i_2 \\ b_{i_1} + b_{i_2} = 2n+1 \end{cases}.$$

Since  $b_{i_2} > i_2$ , it is impossible to have  $i_1 + b_{i_2} = i_2$ .

So, to prove that a system has no repeated blocks is enough to show that  $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n+1 \end{cases}$

or  $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n+1 \end{cases}$  are not satisfied. Also, we show that  $i = \frac{v}{3}$  and  $b_i = \frac{2v}{3}$  is not allowed,

which means that our systems has no short orbits.

For  $n = 8$  and  $n = 12$ , it is easy to see that the Skolem sequences of order  $n$  given by Lemma 3.1 produce simple designs.

For  $n \equiv 0 \pmod{4}$ ,  $n \geq 16$ , let  $S_n$  be the Skolem sequence given by Lemma 3.1. This Skolem sequence is constructed using the hooked Langford sequence  $hL_4^{n-3}$  from [13], Theorem 2, Case 1. Since  $d = 4$ , will use only lines (1)–(7), (14), (8\*), (10\*) and (11\*) in Simpson's Table. Note that  $n - 3 = 9 + 4r$  in Simpson's Table, so  $n = 12 + 4r$  and  $v = 25 + 8r$ ,  $r \geq 1$  in this case. Because we add the pair (1, 1) at the beginning of the Langford sequence  $hL_4^{n-3}$ ,  $a_i$  and  $b_i$  will be shifted to the right by two positions. To make it easier for the reader, we give in Table 1 the  $hL_4^{n-3}$  taken from Simpson's Table and adapted for our case.

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r + 3 - j$	$2r + 7 + j$	$4 + 2j$	$r$
(2)	$r + 2 - j$	$3r + 9 + j$	$2r + 7 + 2j$	$r - 1$
(3)	$6r + 12 - j$	$6r + 17 + j$	$5 + 2j$	$r - 1$
(4)	$5r + 12 - j$	$7r + 18 + j$	$2r + 6 + 2j$	$r$
(5)	$3r + 8$	$7r + 17$	$4r + 9$	-
(6)	$4r + 9$	$8r + 21$	$4r + 12$	-
(7)	$2r + 6$	$6r + 13$	$4r + 7$	-
(14)	$2r + 5$	$6r + 16$	$4r + 11$	-
(8*)	$4r + 11$	$8r + 19$	$4r + 8$	-
(10*)	$4r + 10$	$6r + 15$	$2r + 5$	-
(11*)	$2r + 4$	$6r + 14$	$4r + 10$	-

Table 1:  $hL_4^{n-3}$

So, the base blocks of the cyclic designs produced by Construction 3.2 are  $\{0, 1, 2\}$ ,  $\{0, 2, v - 1\}$ ,  $\{0, 3, v - 2\}$  and  $\{0, i, b_i + 2\}$  for  $i = 4, \dots, n$  and  $i = b_i - a_i$ .

We show first that  $i = \frac{v}{3}$  and  $b_i + 2 = \frac{2v}{3}$  is not allowed in the above system. In the first three base blocks is obvious that  $i \neq \frac{v}{3}$ . For the remaining base blocks we check lines (1) – (7), (14), (8\*), (10\*) and (11\*) in Table 1.

$$\begin{aligned}
\text{Line (1)} & \begin{cases} 4 + 2j = \frac{25+8r}{3} \\ 2r + 7 + j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (2)} & \begin{cases} 2r + 7 + 2j = \frac{25+8r}{3} \\ 3r + 9 + j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (3)} & \begin{cases} 5 + 2j = \frac{25+8r}{3} \\ 6r + 17 + j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (4)} & \begin{cases} 2r + 6 + 2j = \frac{25+8r}{3} \\ 7r + 18 + j = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.
\end{aligned}$$

$$\text{Line (5)} \begin{cases} 4r + 9 = \frac{25+8r}{3} \\ 7r + 17 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (6)} \begin{cases} 4r + 12 = \frac{25+8r}{3} \\ 8r + 21 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (7)} \begin{cases} 4r + 7 = \frac{25+8r}{3} \\ 6r + 13 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (14)} \begin{cases} 4r + 11 = \frac{25+8r}{3} \\ 6r + 16 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (8*)} \begin{cases} 4r + 8 = \frac{25+8r}{3} \\ 8r + 19 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (10*)} \begin{cases} 2r + 5 = \frac{25+8r}{3} \\ 6r + 15 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Line (11*)} \begin{cases} 4r + 10 = \frac{25+8r}{3} \\ 6r + 14 = \frac{2(25+8r)}{3} \end{cases} \Leftrightarrow \emptyset.$$

Therefore, this systems has no short orbits.

Next, we have to check that  $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n + 1 \end{cases}$  or  $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n + 1 \end{cases}$  are not satisfied.

Lines (1) – (1):  $\begin{cases} 4 + 2j_1 + 4 + 2j_2 = 2r + 7 + j_2 \\ 4 + 2j_2 + 2r + 7 + j_1 = 25 + 8r \end{cases} \Leftrightarrow \begin{cases} j_1 = \frac{-2r-15}{3} \\ j_2 = \frac{10r+27}{3} \end{cases}$  which is impossible since  $j_1 \geq 0$  and also integer.

Lines (1) – (2):  $\begin{cases} 4 + 2j_1 + 2r + 7 + 2j_2 = 3r + 9 + j_2 \\ 2r + 7 + 2j_2 + 2r + 7 + j_1 = 25 + 8r \end{cases} \Leftrightarrow \begin{cases} j_1 = \frac{-2r-15}{3} \\ j_2 = r - 2 - 2j_1 \end{cases}$  which is impossible since  $j_1 \geq 0$  and also integer.

Lines (1) – (3):  $\begin{cases} 4 + 2j_1 + 5 + 2j_2 = 6r + 17 + j_2 \\ 5 + 2j_2 + 2r + 7 + j_1 = 25 + 8r \end{cases} \Leftrightarrow \begin{cases} j_1 = j_2 - 5 \\ j_2 = 2r + 9 \end{cases}$  which is impossible since  $j_2 \leq r - 1$ .

Lines (1) – (4):  $\begin{cases} 4 + 2j_1 + 2r + 6 + 2j_2 = 7r + 18 + j_2 \\ 2r + 6 + 2j_2 + 2r + 7 + j_1 = 25 + 8r \end{cases} \Leftrightarrow \begin{cases} j_1 = j_2 + r - 4 \\ j_2 = \frac{3r+16}{3} \end{cases}$  which is impossible since  $j_2 \leq r$ .

$$\text{Lines (1) – (5):} \begin{cases} 4r + 2j + 13 = 7r + 17 \\ j + 6r + 16 = 25 + 8r \end{cases} \Leftrightarrow \emptyset.$$

$$\text{Lines (1) – (6):} \begin{cases} 4r + 2j + 16 = 8r + 21 \\ j + 6r + 19 = 25 + 8r \end{cases} \Leftrightarrow \emptyset.$$

$$\begin{aligned}
\text{Lines (1) - (7): } & \begin{cases} 4r + 2j + 11 = 6r + 13 \\ j + 6r + 14 = 25 + 8r \end{cases} \Leftrightarrow \emptyset. \\
\text{Lines (1) - (14): } & \begin{cases} 4r + 2j + 15 = 6r + 16 \\ j + 6r + 18 = 25 + 8r \end{cases} \Leftrightarrow \emptyset. \\
\text{Lines (1) - (8*): } & \begin{cases} 4r + 2j + 12 = 8r + 19 \\ j + 6r + 15 = 25 + 8r \end{cases} \Leftrightarrow \emptyset. \\
\text{Lines (1) - (10*): } & \begin{cases} 2r + 2j + 9 = 6r + 15 \\ j + 4r + 12 = 25 + 8r \end{cases} \Leftrightarrow \emptyset. \\
\text{Lines (1) - (11*): } & \begin{cases} 4r + 2j + 14 = 6r + 14 \\ j + 6r + 17 = 25 + 8r \end{cases} \Leftrightarrow \emptyset.
\end{aligned}$$

We implemented a program in Mathematica that checks all the pairs of rows in Simpson's table using the above approach. The code for the program and the results can be found in Appendix. From the results, we can easily see that if we check any combination of two lines in Simpson's Table the conditions are not satisfied in almost all of the cases. There are two cases where this conditions are satisfied. The first case is when we check line 3 with line 1, and we get that for  $r = 4 + 3c$ ,  $j_1 = 2c$ , and  $j_2 = 6 + 2c$ ,  $c \geq 2$  the system is not simple. This implies that our system is not simple when  $v = 24c + 57$ ,  $c \geq 2$ . The second case is when we check line 3 with line 2. Here, we get  $r = 5$  and therefore  $v = 59$ . But  $v = 59$  is not congruent to 3 (mod 6). A  $TS_3(59)$  is simple, cyclic, and indecomposable by Theorem 2.1.

For  $n \equiv 1 \pmod{4}$ ,  $n \geq 9$ , let  $S_n$  be the Skolem sequence given by Lemma 3.1. This Skolem sequence is constructed using a  $hL_3^{n-2}$  ([13], Theorem 2, Case 1). Since  $d = 3$ , will use only lines (1) - (6), (14), (7'), (8') and (10') in Simpson's Table. Note that  $n - 2 = 7 + 4r$  in Simpson's Table, so  $n = 9 + 4r$  and  $v = 19 + 8r$  in this case. Because we add the pair (1, 1) at the beginning of the Langford sequence  $hL_3^{n-2}$ ,  $a_i$  and  $b_i$  will be shifted to the right by two positions.

Table 2 gives the  $hL_3^{n-2}$  from Simpson's Table adapted to our case.

So, the base blocks of the cyclic designs produce by Construction 3.2 are  $\{0, 1, 2\}$ ,  $\{0, 2, v-1\}$  and  $\{0, i, b_i + 2\}$  for  $i = 3, \dots, n$ . Using the same argument as before, we show that these designs are simple.

First, we show that  $i = \frac{v}{3}$  and  $b_i + 2 = \frac{2v}{3}$  is not allowed in the above system. In the first two base blocks is obvious that  $i \neq \frac{v}{3}$ . For the remaining base blocks we check lines (1) - (6), (14), (7') and (10') in Table 2.

$$\begin{aligned}
\text{Line (1)} & \begin{cases} 3 + 2j = \frac{19+8r}{3} \\ 2r + 6 + j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (2)} & \begin{cases} 2r + 6 + 2j = \frac{19+8r}{3} \\ 3r + 8 + j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset.
\end{aligned}$$



	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r + 3 - j$	$2r + 6 + j$	$3 + 2j$	$r$
(2)	$r + 2 - j$	$3r + 8 + j$	$2r + 6 + 2j$	$r - 1$
(3)	$6r + 10 - j$	$6r + 14 + j$	$4 + 2j$	$r - 1$
(4)	$5r + 10 - j$	$7r + 15 + j$	$2r + 5 + 2j$	$r$
(5)	$3r + 7$	$7r + 14$	$4r + 7$	-
(6)	$4r + 8$	$8r + 17$	$4r + 9$	-
(14)	$2r + 4$	$6r + 12$	$4r + 8$	-
(7')	$2r + 5$	$6r + 11$	$4r + 6$	-
(10')	$4r + 9$	$6r + 13$	$2r + 4$	-

Table 2:  $hL_3^{n-2}$

$$\begin{aligned}
\text{Line (3)} & \begin{cases} 4 + 2j = \frac{19+8r}{3} \\ 6r + 14 + j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (4)} & \begin{cases} 2r + 5 + 2j = \frac{19+8r}{3} \\ 7r + 15 + j = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (5)} & \begin{cases} 4r + 7 = \frac{19+8r}{3} \\ 7r + 14 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (6)} & \begin{cases} 4r + 9 = \frac{19+8r}{3} \\ 8r + 17 = 2\frac{19+8r}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (14)} & \begin{cases} 4r + 8 = \frac{19+8r}{3} \\ 6r + 12 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (7')} & \begin{cases} 4r + 6 = \frac{19+8r}{3} \\ 6r + 11 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset. \\
\text{Line (10')} & \begin{cases} 2r + 4 = \frac{19+8r}{3} \\ 6r + 13 = \frac{2(19+8r)}{3} \end{cases} \Leftrightarrow \emptyset.
\end{aligned}$$

Next, we have to check that  $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n + 1 \end{cases}$  or  $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n + 1 \end{cases}$  are not satisfied. As with the previous case, the results can be found in Appendix. As before, when we check line 3 and line 1 the conditions are satisfied. But, in this case  $v = 24c + 35$ ,  $c \geq 1$  which is not congruent to 3 (mod 6). So, a  $\text{TS}_3(24c + 35)$  for  $c \geq 1$  is cyclic, simple, and indecomposable by Theorem 2.1. ■

**Lemma 3.3** *For every  $n \equiv 2$  or  $3 \pmod{4}$ ,  $n \geq 7$ , there is a hooked Skolem sequence of order  $n$  in which  $s_1 = s_2 = 1$  and  $s_{2n-1} = s_{2n+1} = 2$ .*

**Proof** For  $n \equiv 2$  or  $3 \pmod{4}$ ,  $n \geq 7$ , take  $hS_n = (1, 1, L_3^{n-2}, 2, 0, 2)$ .

When  $n \equiv 2 \pmod{4}$ , take  $L_3^{n-2}$  ([13], Theorem 1, Case 3). When  $n \equiv 3 \pmod{4}$ , take  $L_3^5 = (6, 7, 3, 4, 5, 3, 6, 4, 7, 5)$  and for  $n \geq 11$  take  $L_3^{n-2}$  (see [2], Theorem 2). ■

We are going to use the following construction to get cyclic  $TS_3(2n+1)$  for  $n \equiv 2$  or  $3 \pmod{4}$ :

**Construction 3.3** [12] *Let  $hS_n = (s_1, s_2, \dots, s_{2n-1}, s_{2n+1})$  be a hooked Skolem sequence of order  $n$  and let  $\{(a_i, b_i) | 1 \leq i \leq n\}$  be the pairs of positions in  $hS_n$  for which  $b_i - a_i = i$ . Then the set  $\mathcal{F} = \{\{0, i, b_i + 1\} | 1 \leq i \leq n\} \pmod{2n+1}$  is a  $(2n+1, 3, 3) - DF$ . Hence, the set of triples in  $\mathcal{F}$  form the base blocks of a cyclic  $TS_3(2n+1)$ .*

Then, we apply Construction 3.3 to the hooked Skolem sequences given by Lemma 3.3 to get cyclic  $TS_3(2n+1)$  for  $n \equiv 2$  or  $3 \pmod{4}$  that are simple and indecomposable.

**Construction 3.4** *Let  $hS_n = (s_1, s_2, \dots, s_{2n-1}, s_{2n+1})$  be a hooked Skolem sequence of order  $n$  given by Lemma 3.3, and let  $\{(a_i, b_i) | 1 \leq i \leq n\}$  be the pairs of positions in  $hS_n$  for which  $b_i - a_i = i$ . Then, the set  $\mathcal{F} = \{\{0, i, b_i + 1\} | 1 \leq i \leq n\} \pmod{2n+1}$  form the base blocks of a cyclic, simple, and indecomposable  $TS_3(2n+1)$ .*

**Example 3.3** *If we apply Construction 3.4 to the hooked Skolem sequence of order 7:  $(1, 1, 6, 7, 3, 4, 5, 3, 6, 4, 7, 5, 2, 0, 2)$  we get the base blocks  $\{\{0, 1, 3\}, \{0, 2, 1\}, \{0, 3, 9\}, \{0, 4, 11\}, \{0, 5, 13\}, \{0, 6, 10\}, \{0, 7, 12\}\} \pmod{15}$ . These base blocks form a cyclic  $TS_3(15)$  by Construction 3.3. We are going to prove next, that this design is indecomposable and simple.*

**Theorem 3.4** *The  $TS_3(6n+3)$ ,  $n \geq 2$ , produced by applying Construction 3.4, are simple.*

**Proof** The proof is similar to Theorem 3.2. Let  $v = 2n+1$ ,  $n \equiv 2$  or  $3 \pmod{4}$ ,  $n \geq 10$ .

For  $n \equiv 2 \pmod{4}$ ,  $n \geq 10$ , let  $hS_n$  be the hooked Skolem sequence given by Lemma 3.3. This hooked Skolem sequence is constructed using the Langford sequence  $L_3^{n-2}$  from [13], Theorem 1, Case 3. Since  $d = 3$ , will use only lines (1) – (4), (6), (9), (11) and (13) in Simpson's Table. Note that  $m = n - 2 = 4r$  in Simpson's Table, so  $n = 4r + 2$ ,  $v = 8r + 5$ ,  $r \geq 2$ ,  $d = 3$ ,  $s = 1$ , in this case. Because we add the pair  $(1, 1)$  at the beginning of the hooked Langford sequence  $hL_3^{n-2}$ ,  $a_i$  and  $b_i$  will be shifted to the right by two positions. To make it easier for the reader, we give in Table 3, the  $L_3^{n-2}$  taken from Simpson's Table and adapted for our case (omit row (1) when  $r = 2$ ).

So, the base blocks of the cyclic designs produce by Construction 3.4 are  $\{0, 1, 3\}$ ,  $\{0, 2, 1\}$  and  $\{0, i, b_i + 2 + 1\}$  for  $i = 3, \dots, n$  and  $i = b_i - a_i$ .

First, we show that  $i = \frac{v}{3}$  and  $b_i + 2 + 1 = \frac{2v}{3}$  is not allowed in the above system. In the first two base blocks is obvious that  $i \neq \frac{v}{3}$ . For the remaining base blocks we check lines (1) – (4), (6), (9), (11) and (13) in Table 3.

Line (1)  $\begin{cases} 4 + 2j = \frac{8r+5}{3} \\ 2r + 5 + j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = \frac{1}{4}$  which is impossible since  $r \geq 2$  and also integer.

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r - j$	$2r + 4 + j$	$4 + 2j$	$r - 3$
(2)	$r + 2 - j$	$3r + 3 + j$	$2r + 1 + 2j$	$r - 1$
(3)	$6r + 1 - j$	$6r + 4 + j$	$3 + 2j$	$r - 2$
(4)	$5r + 2 - j$	$7r + 4 + j$	$2r + 2 + 2j$	$r - 2$
(6)	$2r + 3$	$4r + 3$	$2r$	-
(9)	$3r + 2$	$7r + 3$	$4r + 1$	-
(11)	$2r + 1$	$6r + 3$	$4r + 2$	-
(13)	$2r + 2$	$6r + 2$	$4r$	-

Table 3:  $L_3^{n-2}$

Line (2)  $\begin{cases} 2r + 1 + 2j = \frac{8r+5}{3} \\ 3r + 4 + j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = \frac{1}{2}$  which is impossible since  $r \geq 2$  and also integer.

Line (3)  $\begin{cases} 3 + 2j = \frac{8r+5}{3} \\ 6r + 5 + j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = -\frac{1}{2}$  which is impossible since  $r \geq 2$  and also integer.

Line (4)  $\begin{cases} 2r + 2 + 2j = \frac{8r+5}{3} \\ 7r + 5 + j = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow r = -\frac{3}{4}$  which is impossible since  $r \geq 2$  and also integer.

Line (6)  $\begin{cases} 2r = \frac{8r+5}{3} \\ 4r + 4 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$ . Line (9)  $\begin{cases} 4r + 1 = \frac{8r+5}{3} \\ 7r + 4 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Line (11)  $\begin{cases} 4r + 2 = \frac{8r+5}{3} \\ 6r + 4 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$ . Line (13)  $\begin{cases} 4r = \frac{8r+5}{3} \\ 6r + 3 = \frac{2(8r+5)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Next, we have to show that  $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n + 1 \end{cases}$  or  $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n + 1 \end{cases}$  are not satisfied. The results for this can be found Appendix. As before, when we check line 3 with line 1, the conditions are satisfied. But  $v = 24c + 5$ ,  $c \geq 2$  in this case which is not congruent to 3 (mod 6). So, by Theorem 2.1, there exists a cyclic, simple, and indecomposable  $\text{TS}_3(24c + 5)$  for  $c \geq 2$ .

For  $n \equiv 3 \pmod{4}$ ,  $n \geq 11$ , let  $hS_n$  be the hooked Skolem sequence given by Lemma 3.3. This hooked Skolem sequence is constructed using a  $L_3^{n-2}$  ([2], Theorem 2). Since  $d = 3$  will use only lines (1) – (4), (6) – (10) in [2]. Note that  $m = n - 2 = 4r + 1$ ,  $r \geq 2$ ,  $e = 4$  in [2], so  $n = 4r + 3$  and  $v = 8r + 7$  in this case. Because we add the pair (1, 1) at the beginning of the Langford sequence  $L_3^{n-2}$ ,  $a_i$  and  $b_i$  will be shifted to the right by two positions.

Table 4 gives the  $L_3^{n-2}$  from [2] adapted to our case.

So, the base blocks of the cyclic designs produce by Construction 3.4 are  $\{0, 1, 3\}$ ,  $\{0, 2, 1\}$  and  $\{0, i, b_i + 2 + 1\}$  for  $i = 3, \dots, n$ . Using the same argument as before, we show that these

	$a_i + 2$	$b_i + 2$	$i = b_i - a_i$	$0 \leq j \leq$
(1)	$2r + 2 - j$	$2r + 6 + j$	$4 + 2j$	$r - 2$
(2)	$r + 2 - j$	$3r + 5 + j$	$2r + 3 + 2j$	$r - 2$
(3)	3	$4r + 4$	$4r + 1$	-
(4)	$2r + 4$	$4r + 5$	$2r + 1$	-
(6)	$r + 3$	$5r + 5$	$4r + 2$	-
(7)	$2r + 5$	$6r + 5$	$4r$	-
(8)	$2r + 3$	$6r + 6$	$4r + 3$	-
(9)	$6r + 4 - j$	$6r + 7 + j$	$3 + 2j$	$r - 2$
(10)	$5r + 4 - j$	$7r + 6 + j$	$2r + 2 + 2j$	$r - 2$

Table 4:  $L_3^{n-2}$

designs are simple.

First we show that  $i = \frac{v}{3}$  and  $b_i + 2 + 1 = \frac{2v}{3}$  is not allowed in the above system. In the first two base blocks is obvious that  $i \neq \frac{v}{3}$ . For the remaining base blocks we check lines (1) – (4), (6) – (10) in Table 4.

Line (1)  $\begin{cases} 4 + 2j = \frac{8r+7}{3} \\ 2r + 7 + j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r = \frac{3}{4}$  which is impossible since  $r \geq 2$  and also integer.

Line (2)  $\begin{cases} 2r + 3 + 2j = \frac{8r+7}{3} \\ 3r + 6 + j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow r = \frac{1}{2}$  which is impossible since  $r \geq 2$  and also integer.

Line (3)  $\begin{cases} 4r + 1 = \frac{8r+7}{3} \\ 4r + 5 = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ . Line (4)  $\begin{cases} 2r + 1 = \frac{8r+7}{3} \\ 4r + 6 = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Line (6)  $\begin{cases} 4r + 2 = \frac{8r+7}{3} \\ 5r + 6 = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ . Line (7)  $\begin{cases} 4r = \frac{8r+7}{3} \\ 6r + 6 = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Line (8)  $\begin{cases} 4r + 3 = \frac{8r+7}{3} \\ 6r + 7 = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ . Line (9)  $\begin{cases} 3 + 2j = \frac{8r+7}{3} \\ 6r + 8 + j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Line (10)  $\begin{cases} 2r + 2 + 2j = \frac{8r+7}{3} \\ 7r + 7 + j = \frac{2(8r+7)}{3} \end{cases} \Leftrightarrow \emptyset$ .

Next, we have to check that  $\begin{cases} i_1 + i_2 = b_{i_2} \\ b_{i_1} + i_2 = 2n + 1 \end{cases}$  or  $\begin{cases} i_1 + i_2 = b_{i_1} \\ b_{i_2} + i_1 = 2n + 1 \end{cases}$  are not satisfied. The results for this can be found in Appendix. Here, for  $v = 3c - 1$ ,  $c \geq 4$  and for  $v = 55$  the conditions are satisfied but these orders are not congruent to 3 (mod 6). Therefore, by Theorem 2.1, there exists cyclic, simple, and indecomposable  $\text{TS}_3(3c - 1)$  for  $c \geq 4$  and cyclic, simple, and indecomposable  $\text{TS}_3(55)$ . ■

## 4 Indecomposable Three-fold Cyclic Triple Systems

In this section, we prove that the Constructions 3.2 and 3.4 produce indecomposable three-fold triple systems for  $v \equiv 3 \pmod{6}$ ,  $v \geq 15$ .

**Theorem 4.1** *The  $TS_3(v)$  produced by Constructions 3.2 and 3.4 are indecomposable for every  $v \equiv 3 \pmod{6}$ ,  $v \geq 15$ .*

**Proof** Suppose that  $v \equiv 3 \pmod{6}$  and write  $v = 2n + 1$ ,  $n \equiv 0$  or  $1 \pmod{4}$ ,  $n \geq 8$ .

Now, for an  $TS_3(2n + 1)$  to be decomposable, there must be a Steiner triple system  $STS(2n + 1)$  inside the  $TS_3(2n + 1)$ .

If  $2n + 1 \equiv 3 \pmod{6}$ , let  $\{x_i, x_j, x_k\}$  be a triple using symbols from  $N_{2n+1} = \{0, 1, \dots, 2n\}$ . Let  $d_{ij} = \min\{|x_i - x_j|, 2n + 1 - |x_i - x_j|\}$  be the difference between  $x_i$  and  $x_j$ . An  $STS(2n + 1)$  on  $N_{2n+1}$  must have a set of triples with the property that each difference  $d$ ,  $1 \leq d \leq n$ , occurs exactly  $2n + 1$  times. Assume there is an  $STS(2n + 1)$  inside our  $TS_3(2n + 1)$  and let  $f_\alpha$  be the number of triples inside the  $STS(2n + 1)$  which are a cyclic shift of  $\{0, \alpha, b_\alpha\}$ .

It is enough to look at the first two base blocks of our  $TS_3(2n + 1)$ . These are  $\{0, 1, 2\} \pmod{2n + 1}$  and  $\{0, 2, 2n\} \pmod{2n + 1}$ . Then the existence of an  $STS(2n + 1)$  inside our  $TS_3(2n + 1)$  requires that the equation  $2f_1 + f_2 = 2n + 1$  must have a solution in nonnegative integers (we need the difference 1 to occur exactly  $2n + 1$  times).

**Case 1:  $f_1 = 1$**

Suppose we choose one block from the orbit  $\{0, 1, 2\} \pmod{2n + 1}$ . Since this orbit uses the difference 1 twice and the difference 2, and the orbit  $\{0, 2, 2n\} \pmod{2n + 1}$  uses the differences 1, 2 and 3, whenever we pick one block from the first orbit we cannot choose three blocks from the second orbit (i.e., those blocks where the pairs  $(0, 1)$ ,  $(0, 2)$  and  $(1, 2)$  are included). So, we just have  $2n - 2$  blocks in the second orbit to choose from. But we need  $2n - 1$  blocks from the second orbit in order to cover difference 1 exactly  $2n + 1$  times.

Therefore, we have no solution in this case.

**Case 2:  $f_1 = 2$**

Since  $f_2 = \frac{2n-3}{2}$  is not an integer, we have no solution in this case.

**Case 3:  $f_1 = 3, 5, \dots, n$  (or  $n - 1$ )**

Similar to Case 1. So, there is no solution in this case.

**Case 4:  $f_1 = 4, 6, \dots, n$  (or  $n - 1$ )**

Similar to Case 2. So, there is no solution in this case.

**Case 5:  $f_1 = 0$**

Note that our cyclic  $TS_3(v)$  has no short orbits (Theorem 3.2), while a cyclic  $STS(v)$  will have a short orbit. Therefore, if a design exists inside our  $TS_3(v)$ , that design is not cyclic.

Now, we choose no block from the first orbit and all the blocks in the second orbit (i.e.,  $f_1 = 0$ ,  $f_2 = 2n + 1$ ). Therefore differences 1, 2 and 3 are all covered each exactly  $2n + 1$  in the  $STS(v)$ . From the remaining  $n - 2$  orbits  $\{0, i, b_i\}$ ,  $i \geq 3$  there will be two or three orbits which will use differences 2 and 3. Since differences 1, 2 and 3 are already covered, we cannot choose any block from those orbits that uses these three differences. So, we are

left with  $n - 4$  or  $n - 5$  orbits to choose from. We need to cover differences  $4, 5, \dots, n$  ( $n - 3$  differences) each exactly  $v = 2n + 1$  times.

We form a system of  $n - 3$  equations with  $n - 4$  or  $n - 5$  unknowns in the following way: when a difference appears in different orbits, the sum of the blocks that we choose from each orbit has to equal  $v$ , i.e., if difference 4 appears in  $\{0, 5, b_5\}$ ,  $\{0, 7, b_7\}$  and  $\{0, 10, b_{10}\}$  we have  $f_5 + f_7 + f_{10} = v$  or if difference 4 appears in  $\{0, 7, b_7\}$  twice and in  $\{0, 9, b_9\}$  once, we have  $2f_7 + f_9 = v$ . The system that we form has two or three entries in each row non-zero while all the others entries will equal zero. The rows in the system can be rearranged so that we get an upper triangular matrix. Therefore, the system of equations is non-singular and it has the unique solution  $f_{i_1} = f_{i_2} = \dots = f_{i_k} = v$  for some  $4 \leq i_1, i_2, \dots, i_k \leq n$  and  $f_{j_1} = f_{j_2} = \dots = f_{j_k} = 0$  for some  $4 \leq j_1, j_2, \dots, j_k \leq n$ . But this implies that the  $\text{STS}(v)$  inside our  $\text{TS}_3(v)$  is cyclic, which is impossible.

Therefore, we have no solution in this case. It follows that our  $\text{TS}_3(2n + 1)$  is indecomposable.

Now, suppose that  $v = 2n + 1$ ,  $n \equiv 2$  or  $3 \pmod{4}$ ,  $n \geq 7$ . Let  $f_\alpha$  be the number of triples inside the  $\text{STS}(2n + 1)$  which are a cyclic shift of  $\{0, \alpha, b_\alpha + 1\}$ . Using the same argument as before, it is easy to show that the equation  $2f_2 + f_1 = 2n + 1$  has no solution. Therefore our  $\text{TS}_3(2n + 1)$  is indecomposable. ■

## 5 Cyclic, Simple, and Indecomposable Three-fold Triple Systems

**Theorem 5.1** *There exists cyclic, simple, and indecomposable three-fold triple systems,  $\text{TS}_3(v)$ , for every  $v \equiv 1 \pmod{2}$ ,  $v \geq 5$ ,  $v \neq 9$  and  $v \neq 24c + 57$ ,  $c \geq 2$ .*

**Proof** Let  $v \equiv 1$  or  $5 \pmod{6}$  and take the base blocks  $\{0, \alpha, -\alpha\} \pmod{v} | \alpha = 0, 1, \dots, \frac{1}{2}(v - 1)$ . By Theorem 2.1, these will be the base blocks of a cyclic, simple, and indecomposable three-fold triple system.

Let  $v \equiv 3 \pmod{6}$ , and write  $v = 2n + 1$ ,  $n \equiv 0$  or  $1 \pmod{4}$ ,  $n \geq 8$ . Apply Construction 3.2 to the Skolem sequence of order  $n$  given by Lemma 3.1. These designs are cyclic by Construction 3.1, simple for all  $v$  except  $v = 24c + 57$ ,  $c \geq 2$  by Theorem 3.2 and indecomposable by Theorem 4.1.

Let  $v \equiv 3 \pmod{6}$ , and write  $v = 2n + 1$ ,  $n \equiv 2$  or  $3 \pmod{4}$ ,  $n \geq 7$ . Apply Construction 3.4 to the hooked Skolem sequence of order  $n$  given by Lemma 3.3. These designs are cyclic by Construction 3.3, simple by Theorem 3.4 and indecomposable by Theorem 4.1. ■

## 6 Conclusion and Open Problems

We constructed, using Skolem-type sequences, three-fold triple systems having all the properties of being cyclic, simple and indecomposable, for  $v \equiv 3 \pmod{6}$  except for  $v = 9$  and  $v = 24c + 57$ ,  $c \geq 2$ . Our results, together with Kramer's results [9], completely solve

the problem of finding three-fold triple systems having three properties: cyclic, simple, and indecomposable with some possible exceptions for  $v = 9$  and  $v = 24c + 7$ ,  $c \geq 2$ .

In our approach of finding cyclic, simple, and indecomposable three-fold triple systems, proving the simplicity of the designs was a tedious and long task. Another approach that we tried was in constructing three disjoint (i.e. no two pairs in the same positions) sequences of order  $n$  and taking the base blocks  $\{0, i, b_i + n\}$ ,  $i = 1, 2, \dots, n$ . These base blocks form a cyclic  $TS_3(6n + 1)$ . Also, someone can take three disjoint hooked sequences of order  $n$  and taking the base blocks  $\{0, i, b_i + n\}$ ,  $i = 1, 2, \dots, n$  together with the short orbit  $\{0, \frac{v}{3}, \frac{2v}{3}\}$ . These base blocks form a cyclic  $TS_3(6n + 3)$ .

**Example 6.1** For  $n = 5$ , take the three disjoint hooked sequences of order  $n$ :

$$\begin{aligned} &(1, 1, 4, 1, 1, 0, 4, 2, 3, 2, 0, 3) \\ &(2, 3, 2, 3, 3, 0, 3, 4, 1, 1, 0, 4) \\ &(4, 5, 5, 5, 4, 0, 5, 5, 5, 2, 0, 2) \end{aligned}$$

Then the base blocks  $\{0, 1, 7\}, \{0, 1, 10\}, \{0, 1, 15\}, \{0, 2, 8\}, \{0, 2, 15\}, \{0, 2, 17\}, \{0, 3, 10\}, \{0, 3, 12\}, \{0, 3, 17\}, \{0, 4, 10\}, \{0, 4, 17\}, \{0, 4, 12\}, \{0, 5, 12\}, \{0, 5, 13\}, \{0, 5, 14\}, \{0, 11, 22\}$  are the base blocks of a cyclic, simple, and indecomposable  $TS_3(33)$ .

The simplicity is easy to prove here since the three hooked sequences that we used share no pairs in the same positions. On the other hand, on first inspection, to prove the indecomposability of such designs appears to be more difficult. Also, someone needs to find three disjoint such sequences for all admissible orders  $n$ .

**Problem 6.1** Can the above approach of finding cyclic, simple, and indecomposable three-fold triple systems of order  $v$  be generalized for all admissible orders?

**Problem 6.2** Are there cyclic, simple, and indecomposable  $TS_3(24c + 57)$ ,  $c \geq 2$ ?

**Problem 6.3** Are there cyclic, simple, and indecomposable designs for  $\lambda \geq 4$  and all admissible orders?

**Problem 6.4** For  $\lambda \geq 3$  what is the spectrum of those  $v$  for which there exists a cyclically indecomposable but decomposable cyclic  $TS_\lambda(v)$ ?

**Example 6.2** Let  $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ . Then the blocks of a cyclic  $TS_3(9)$  are:

$$\begin{aligned} &\{0, 1, 2\}, \{0, 2, 7\}, \{0, 3, 6\}, \{0, 4, 8\} \\ &\{1, 2, 3\}, \{1, 3, 8\}, \{1, 4, 7\}, \{1, 5, 0\} \\ &\{2, 3, 4\}, \{2, 4, 0\}, \{2, 5, 8\}, \{2, 6, 1\} \\ &\{3, 4, 5\}, \{3, 5, 1\}, \{3, 6, 0\}, \{3, 7, 2\} \\ &\{4, 5, 6\}, \{4, 6, 2\}, \{4, 7, 1\}, \{4, 8, 3\} \\ &\{5, 6, 7\}, \{5, 7, 3\}, \{5, 8, 2\}, \{5, 0, 4\} \end{aligned}$$

$\{6, 7, 8\}, \{6, 8, 4\}, \{6, 0, 3\}, \{6, 1, 5\}$   
 $\{7, 8, 0\}, \{7, 0, 5\}, \{7, 1, 4\}, \{7, 2, 6\}$   
 $\{8, 0, 1\}, \{8, 1, 6\}, \{8, 2, 5\}, \{8, 3, 7\}$

*This design is cyclic, simple, and is decomposable. The blocks  $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{1, 3, 8\}, \{4, 6, 2\}, \{7, 0, 5\}, \{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}, \{0, 4, 8\}, \{3, 7, 2\}, \{6, 1, 5\}$  form an STS(9). The design is cyclically indecomposable since no CSTS(9) exists.*

## References

- [1] D. Archdeacon and J. Dinitz, Indecomposable triple systems exist for all  $\lambda$ , *Discrete Math.* **113** (1993), 1-6.
- [2] J.C. Bermond, A. E. Brouwer and A. Germa, Systemes de triplets et differences associees, *Colloq. CRNS, Problemes Combinatoires et Theorie des Graphes* (1976), Orsay, 35-38.
- [3] C. J. Colbourn and M. J. Colbourn, The computational complexity of decomposing block designs, *Ann. Discrete Math.* **26** (1985), 345-350.
- [4] M. J. Colbourn and C. J. Colbourn, Cyclic block designs with block size 3, *European J. Combin.* **2** (1981), 21-26.
- [5] C. J. Colbourn and A. Rosa, Indecomposable triple systems with  $\lambda = 4$ , *Studia Sci. Math. Hungar.* **20** (1985), 139-144.
- [6] M. Dehon, On the existence of 2-designs  $S_\lambda(2, 3, v)$  without repeated blocks, *Discrete Math.* **43** (1983), 155-171.
- [7] J. H. Dinitz, Indecomposable triple systems with  $\lambda = 6$ , *J. Combin. Math. Combin. Comput.* **5** (1989), 139-142.
- [8] M. Gruttmuller, R. Rees and N. Shalaby, Cyclically indecomposable triple systems that are decomposable, *J. Combin. Math. Combin. Comput.* **63** (2007), 103-122.
- [9] E. S. Kramer, Indecomposable triple systems, *Discrete Math.* **8** (1974), 173-180.
- [10] S. Milici, Indecomposable  $S_6(2, 3, v)$ 's, *J. Combin. Theory Ser. A* **56** (1991), 16-26.
- [11] R. Rees and N. Shalaby, Simple and indecomposable two-fold cyclic triple systems from Skolem sequences, *J. Combin. Des.* **8** (2000), 402-410.
- [12] D. Silvesan and N. Shalaby, Cyclic block designs with block size 3 from Skolem-type sequences, *Des. Codes and Cryptog.* **63** (2012), 345-355.



- [13] J. E. Simpson, Langford sequences, perfect and hooked, *Discrete Math.* **44** (1983), 97-104.
- [14] H. Shen, Indecomposable of triple systems without repeated blocks, *Ars. Combin.* **33** (1992), 305-310.
- [15] D. R. Stinson and W.D. Wallis, Two-fold triple systems without repeated blocks, *Discrete Math.* **47** (1983), 125-128.
- [16] G. X. Wang, Existence of simple cyclic triple systems, *J. Shanghai Jiaotong Univ.* **25**, No 5, (1991), 95-100.
- [17] X. Zhang, Constructions for indecomposable simple  $(v, k, \lambda)$ -BIBDs, *Discrete Math.* **156** (1996), 317-322.
- [18] X. Zhang, Indecomposable triple systems with  $\lambda = 5$ , *J. Combin. Math. Combin. Comput.*, to appear.

## 7 Appendix A: Mathematica Program

```

Ltable = {1, 2, 3, 4};
Ltable[[1]] = {{2*r + 3 - j, 2*r + 7 + j, 4 + 2*j, r},
  {r + 2 - j, 3*r + 9 + j, 2*r + 7 + 2*j, r - 1},
  {6*r + 12 - j, 6*r + 17 + j, 5 + 2*j, r - 1},
  {5*r + 12 - j, 7*r + 18 + j, 2*r + 6 + 2*j, r},
  {3*r + 8, 7*r + 17, 4*r + 9, 0},
  {4*r + 9, 8*r + 21, 4*r + 12, 0},
  {2*r + 6, 6*r + 13, 4*r + 7, 0},
  {2*r + 5, 6*r + 16, 4*r + 11, 0},
  {4*r + 11, 8*r + 19, 4*r + 8, 0},
  {4*r + 10, 6*r + 15, 2*r + 5, 0},
  {2*r + 4, 6*r + 14, 4*r + 10, 0}
};
Ltable[[2]] = {{2*r + 3 - j, 2*r + 6 + j, 3 + 2*j, r},
  {r + 2 - j, 3*r + 8 + j, 2*r + 6 + 2*j, r - 1},
  {6*r + 10 - j, 6*r + 14 + j, 4 + 2*j, r - 1},
  {5*r + 10 - j, 7*r + 15 + j, 2*r + 5 + 2*j, r},
  {3*r + 7, 7*r + 14, 4*r + 7, 0},
  {4*r + 8, 8*r + 17, 4*r + 9, 0},
  {2*r + 4, 6*r + 12, 4*r + 8, 0},
  {2*r + 5, 6*r + 11, 4*r + 6, 0},
  {4*r + 9, 6*r + 13, 2*r + 4, 0}
};
Ltable[[3]] = {{2*r - j, 2*r + 4 + j, 4 + 2*j, r - 3},
  {r + 2 - j, 3*r + 3 + j, 2*r + 1 + 2*j, r - 1},
  {6*r + 1 - j, 6*r + 4 + j, 3 + 2*j, r - 2},
  {5*r + 2 - j, 7*r + 4 + j, 2*r + 2 + 2*j, r - 2},
  {2*r + 3, 4*r + 3, 2*r, 0},
  {3*r + 2, 7*r + 3, 4*r + 1, 0},
  {2*r + 1, 6*r + 3, 4*r + 2, 0},
  {2*r + 2, 6*r + 2, 4*r, 0}
};
Ltable[[4]] = {{2*r + 2 - j, 2*r + 6 + j, 4 + 2*j, r - 2},
  {r + 2 - j, 3*r + 5 + j, 2*r + 3 + 2*j, r - 2},
  {3, 4*r + 4, 4*r + 1, 0},
  {2*r + 4, 4*r + 5, 2*r + 1, 0},

```

```

{r + 3, 5*r + 5, 4*r + 2, 0},
{2*r + 5, 6*r + 5, 4*r, 0},
{2*r + 3, 6*r + 6, 4*r + 3, 0},
{6*r + 4 - j, 6*r + 7 + j, 3 + 2*j, r - 2},
{5*r + 4 - j, 7*r + 6 + j, 2*r + 2 + 2*j, r - 2}
};

eqns = {i1 + i2 == bi2, bi1 + i2 == 2*n + 1};
thmMapping = {"Theorem 4.2 Case n = 4k", "Theorem 4.2 Case n = 4k+1",
"Theorem 4.4 Case n = 4k+2", "Theorem 4.4 Case n = 4k+3"};
nMapping = {12 + 4*r, 9 + 4*r, 4*r + 2, 4*r + 3};
rbound = {1, 0, 2, 2};
summary = {};

Off[Solve::svars];

Print["System of Equations:"];
Print[eqns[[1]]];
Print[eqns[[2]]];

For[thm = 1, thm <= 4, thm++,
Print["*****"];
Print[thmMapping[[thm]]];
For[i = 1, i <= Length[Ltable[[thm]], i++,
For[k = 1, k <= Length[Ltable[[thm]], k++,
teq1 =
eqns[[1]] /. {i1 -> Ltable[[thm, i, 3]] /. j -> j1,
i2 -> Ltable[[thm, k, 3]] /. j -> j2};
teq2 =
eqns[[2]] /. {i2 -> Ltable[[thm, k, 3]] /. j -> j2,
n -> nMapping[[thm]]};
If[thm == 1 || thm == 2,
teq1 = teq1 /. bi2 -> Ltable[[thm, k, 2]] /. j -> j2;
teq2 = teq2 /. bi1 -> Ltable[[thm, i, 2]] /. j -> j1;
teq1 = teq1 /. bi2 -> Ltable[[thm, k, 2]] + 1 /. j -> j2;
teq2 = teq2 /. bi1 -> Ltable[[thm, i, 2]] + 1 /. j -> j1;
];
Print["Lines " <> ToString[i] <> " and " <> ToString[k]];
Print[teq1];
Print[teq2];
sol =
Solve[teq1 && teq2 && j1 >= 0 && j2 >= 0 && r >= rbound[[thm]] &&
j1 <= Ltable[[thm, i, 4]] && j2 <= Ltable[[thm, k, 4]], {j1,
j2, r}, Integers];
If[Length[sol] > 0,
summary = Append[summary, {thm, i, k, sol}];
Print[sol];
Print["No solutions"];
];
Print["-----"];
];
];
];
Print["*****"];
Print["Exception Cases:"];
Print["Theorem, First Line, Second Line, Solution"];
For[i = 1, i <= Length[summary], i++,
Print[thmMapping[[summary[[i, 1]]]] <> ", " <>
ToString[summary[[i, 2]]] <> ", " <> ToString[summary[[i, 3]]] <>
", " <> ToString[summary[[i, 4]]]];
];

```

## 8 Appendix B: Program Results

System of Equations:

$$i1+i2==bi2$$

$$bi1+i2==1+2 \ n$$

\*\*\*\*\*

Theorem 4.2 Case  $n = 4k$

Lines 1 and 1

$$8+2 \ j1+2 \ j2==7+j2+2 \ r$$

$$11+j1+2 \ j2+2 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 2

$$11+2 \ j1+2 \ j2+2 \ r==9+j2+3 \ r$$

$$14+j1+2 \ j2+4 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 3

$$9+2 \ j1+2 \ j2==17+j2+6 \ r$$

$$12+j1+2 \ j2+2 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 4

$$10+2 \ j1+2 \ j2+2 \ r==18+j2+7 \ r$$

$$13+j1+2 \ j2+4 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 5

$$13+2 \ j1+4 \ r==17+7 \ r$$

$$16+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 6

$$16+2 \ j1+4 \ r==21+8 \ r$$

$$19+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 7

$$11+2 \ j1+4 \ r==13+6 \ r$$

$$14+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 8

$$15+2 \ j1+4 \ r==16+6 \ r$$

$$18+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 9

$$12+2 \ j1+4 \ r==19+8 \ r$$

$$15+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 10

$$9+2 \ j1+2 \ r==15+6 \ r$$

$$12+j1+4 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 1 and 11

$$14+2 \ j1+4 \ r==14+6 \ r$$

$$17+j1+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 2 and 1

$11+2 j_1+2 j_2+2 r==7+j_2+2 r$   
 $13+j_1+2 j_2+3 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 2  
 $14+2 j_1+2 j_2+4 r==9+j_2+3 r$   
 $16+j_1+2 j_2+5 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 3  
 $12+2 j_1+2 j_2+2 r==17+j_2+6 r$   
 $14+j_1+2 j_2+3 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 4  
 $13+2 j_1+2 j_2+4 r==18+j_2+7 r$   
 $15+j_1+2 j_2+5 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 5  
 $16+2 j_1+6 r==17+7 r$   
 $18+j_1+7 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 6  
 $19+2 j_1+6 r==21+8 r$   
 $21+j_1+7 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 7  
 $14+2 j_1+6 r==13+6 r$   
 $16+j_1+7 r==1+2 (12+4 r)$

No solutions  
 -----  
 Lines 2 and 8  
 $18+2 j_1+6 r==16+6 r$   
 $20+j_1+7 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 9  
 $15+2 j_1+6 r==19+8 r$   
 $17+j_1+7 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 10  
 $12+2 j_1+4 r==15+6 r$   
 $14+j_1+5 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 2 and 11  
 $17+2 j_1+6 r==14+6 r$   
 $19+j_1+7 r==1+2 (12+4 r)$   
 No solutions  
 -----  
 Lines 3 and 1  
 $9+2 j_1+2 j_2==7+j_2+2 r$   
 $21+j_1+2 j_2+6 r==1+2 (12+4 r)$   
 $\{\{j_1 \rightarrow \text{ConditionalExpression}[2 C[1],$   
 $\quad C[1] \setminus [\text{Element}] \text{Integers} \&\& C[1] \geq 2],$   
 $\quad j_2 \rightarrow \text{ConditionalExpression}[6+2 C[1],$   
 $\quad C[1] \setminus [\text{Element}] \text{Integers} \&\& C[1] \geq 2],$   
 $\quad r \rightarrow \text{ConditionalExpression}[4+3 C[1],$   
 $\quad C[1] \setminus [\text{Element}] \text{Integers} \&\& C[1] \geq 2]\}\}$   
 -----  
 Lines 3 and 2  
 $12+2 j_1+2 j_2+2 r==9+j_2+3 r$   
 $24+j_1+2 j_2+8 r==1+2 (12+4 r)$

{{j1->1,j2->0,r->5}}

-----

Lines 3 and 3

10+2 j1+2 j2==17+j2+6 r

22+j1+2 j2+6 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 4

11+2 j1+2 j2+2 r==18+j2+7 r

23+j1+2 j2+8 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 5

14+2 j1+4 r==17+7 r

26+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 6

17+2 j1+4 r==21+8 r

29+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 7

12+2 j1+4 r==13+6 r

24+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 8

16+2 j1+4 r==16+6 r

28+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 9

13+2 j1+4 r==19+8 r

25+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 10

10+2 j1+2 r==15+6 r

22+j1+8 r==1+2 (12+4 r)

No solutions

-----

Lines 3 and 11

15+2 j1+4 r==14+6 r

27+j1+10 r==1+2 (12+4 r)

No solutions

-----

Lines 4 and 1

10+2 j1+2 j2+2 r==7+j2+2 r

22+j1+2 j2+7 r==1+2 (12+4 r)

No solutions

-----

Lines 4 and 2

13+2 j1+2 j2+4 r==9+j2+3 r

25+j1+2 j2+9 r==1+2 (12+4 r)

No solutions

-----

Lines 4 and 3

11+2 j1+2 j2+2 r==17+j2+6 r

23+j1+2 j2+7 r==1+2 (12+4 r)

No solutions

-----

Lines 4 and 4

12+2 j1+2 j2+4 r==18+j2+7 r

$24+j_1+2 \ j_2+9 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 5

$15+2 \ j_1+6 \ r==17+7 \ r$

$27+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 6

$18+2 \ j_1+6 \ r==21+8 \ r$

$30+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 7

$13+2 \ j_1+6 \ r==13+6 \ r$

$25+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 8

$17+2 \ j_1+6 \ r==16+6 \ r$

$29+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 9

$14+2 \ j_1+6 \ r==19+8 \ r$

$26+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 10

$11+2 \ j_1+4 \ r==15+6 \ r$

$23+j_1+9 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 4 and 11

$16+2 \ j_1+6 \ r==14+6 \ r$

$28+j_1+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 1

$13+2 \ j_2+4 \ r==7+j_2+2 \ r$

$21+2 \ j_2+7 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 2

$16+2 \ j_2+6 \ r==9+j_2+3 \ r$

$24+2 \ j_2+9 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 3

$14+2 \ j_2+4 \ r==17+j_2+6 \ r$

$22+2 \ j_2+7 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 4

$15+2 \ j_2+6 \ r==18+j_2+7 \ r$

$23+2 \ j_2+9 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 5

$18+8 \ r==17+7 \ r$

$26+11 \ r==1+2 \ (12+4 \ r)$

No solutions

Lines 5 and 6

True	No solutions
29+11 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 2
-----	19+2 j2+6 r==9+j2+3 r
Lines 5 and 7	28+2 j2+10 r==1+2 (12+4 r)
16+8 r==13+6 r	No solutions
24+11 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 3
-----	17+2 j2+4 r==17+j2+6 r
Lines 5 and 8	26+2 j2+8 r==1+2 (12+4 r)
20+8 r==16+6 r	No solutions
28+11 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 4
-----	18+2 j2+6 r==18+j2+7 r
Lines 5 and 9	27+2 j2+10 r==1+2 (12+4 r)
17+8 r==19+8 r	No solutions
25+11 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 5
-----	21+8 r==17+7 r
Lines 5 and 10	30+12 r==1+2 (12+4 r)
14+6 r==15+6 r	No solutions
22+9 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 6
-----	24+8 r==21+8 r
Lines 5 and 11	33+12 r==1+2 (12+4 r)
19+8 r==14+6 r	No solutions
27+11 r==1+2 (12+4 r)	-----
No solutions	Lines 6 and 7
-----	19+8 r==13+6 r
Lines 6 and 1	28+12 r==1+2 (12+4 r)
16+2 j2+4 r==7+j2+2 r	No solutions
25+2 j2+8 r==1+2 (12+4 r)	-----

Lines 6 and 8

$$23+8 \ r==16+6 \ r$$

$$32+12 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 6 and 9

$$20+8 \ r==19+8 \ r$$

$$29+12 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 6 and 10

$$17+6 \ r==15+6 \ r$$

$$26+10 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 6 and 11

$$22+8 \ r==14+6 \ r$$

$$31+12 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 1

$$11+2 \ j2+4 \ r==7+j2+2 \ r$$

$$17+2 \ j2+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 2

$$14+2 \ j2+6 \ r==9+j2+3 \ r$$

$$20+2 \ j2+8 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 3

$$12+2 \ j2+4 \ r==17+j2+6 \ r$$

$$18+2 \ j2+6 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 4

$$13+2 \ j2+6 \ r==18+j2+7 \ r$$

$$19+2 \ j2+8 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 5

$$16+8 \ r==17+7 \ r$$

$$22+10 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 6

$$19+8 \ r==21+8 \ r$$

$$25+10 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 7

$$14+8 \ r==13+6 \ r$$

$$20+10 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 8

$$18+8 \ r==16+6 \ r$$

$$24+10 \ r==1+2 \ (12+4 \ r)$$

No solutions

-----

Lines 7 and 9

$$15+8 \ r==19+8 \ r$$

$$21+10 \ r==1+2 \ (12+4 \ r)$$

No solutions



```

-----
Lines 7 and 10
12+6 r==15+6 r
18+8 r==1+2 (12+4 r)
No solutions
-----
Lines 7 and 11
17+8 r==14+6 r
23+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 1
15+2 j2+4 r==7+j2+2 r
20+2 j2+6 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 2
18+2 j2+6 r==9+j2+3 r
23+2 j2+8 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 3
16+2 j2+4 r==17+j2+6 r
21+2 j2+6 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 4
17+2 j2+6 r==18+j2+7 r
22+2 j2+8 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 5

```

```

20+8 r==17+7 r
25+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 6
23+8 r==21+8 r
28+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 7
18+8 r==13+6 r
23+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 8
22+8 r==16+6 r
27+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 9
True
24+10 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 10
16+6 r==15+6 r
21+8 r==1+2 (12+4 r)
No solutions
-----
Lines 8 and 11
21+8 r==14+6 r
26+10 r==1+2 (12+4 r)

```

No solutions

-----

Lines 9 and 1

$12+2 \ j2+4 \ r==7+j2+2 \ r$

$23+2 \ j2+8 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 2

$15+2 \ j2+6 \ r==9+j2+3 \ r$

$26+2 \ j2+10 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 3

$13+2 \ j2+4 \ r==17+j2+6 \ r$

$24+2 \ j2+8 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 4

$14+2 \ j2+6 \ r==18+j2+7 \ r$

$25+2 \ j2+10 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 5

$17+8 \ r==17+7 \ r$

$28+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 6

$20+8 \ r==21+8 \ r$

$31+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 7

$15+8 \ r==13+6 \ r$

$26+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 8

$19+8 \ r==16+6 \ r$

$30+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 9

$16+8 \ r==19+8 \ r$

$27+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 10

$13+6 \ r==15+6 \ r$

$24+10 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 9 and 11

$18+8 \ r==14+6 \ r$

$29+12 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 10 and 1

$9+2 \ j2+2 \ r==7+j2+2 \ r$

$19+2 \ j2+6 \ r==1+2 \ (12+4 \ r)$

No solutions

-----

Lines 10 and 2

$12+2 \ j2+4 \ r==9+j2+3 \ r$

22+2 j2+8 r==1+2 (12+4 r)	-----
No solutions	Lines 10 and 9
-----	13+6 r==19+8 r
Lines 10 and 3	23+10 r==1+2 (12+4 r)
10+2 j2+2 r==17+j2+6 r	No solutions
20+2 j2+6 r==1+2 (12+4 r)	-----
No solutions	Lines 10 and 10
-----	10+4 r==15+6 r
Lines 10 and 4	20+8 r==1+2 (12+4 r)
11+2 j2+4 r==18+j2+7 r	No solutions
21+2 j2+8 r==1+2 (12+4 r)	-----
No solutions	Lines 10 and 11
-----	15+6 r==14+6 r
Lines 10 and 5	25+10 r==1+2 (12+4 r)
14+6 r==17+7 r	No solutions
24+10 r==1+2 (12+4 r)	-----
No solutions	Lines 11 and 1
-----	14+2 j2+4 r==7+j2+2 r
Lines 10 and 6	18+2 j2+6 r==1+2 (12+4 r)
17+6 r==21+8 r	No solutions
27+10 r==1+2 (12+4 r)	-----
No solutions	Lines 11 and 2
-----	17+2 j2+6 r==9+j2+3 r
Lines 10 and 7	21+2 j2+8 r==1+2 (12+4 r)
12+6 r==13+6 r	No solutions
22+10 r==1+2 (12+4 r)	-----
No solutions	Lines 11 and 3
-----	15+2 j2+4 r==17+j2+6 r
Lines 10 and 8	19+2 j2+6 r==1+2 (12+4 r)
True	No solutions
26+10 r==1+2 (12+4 r)	-----
No solutions	Lines 11 and 4

16+2 j2+6 r==18+j2+7 r	No solutions
20+2 j2+8 r==1+2 (12+4 r)	-----
No solutions	Lines 11 and 11
-----	20+8 r==14+6 r
Lines 11 and 5	24+10 r==1+2 (12+4 r)
19+8 r==17+7 r	No solutions
23+10 r==1+2 (12+4 r)	-----
No solutions	*****
-----	Theorem 4.2 Case n = 4k+1
Lines 11 and 6	Lines 1 and 1
22+8 r==21+8 r	6+2 j1+2 j2==6+j2+2 r
26+10 r==1+2 (12+4 r)	9+j1+2 j2+2 r==1+2 (9+4 r)
No solutions	No solutions
-----	-----
Lines 11 and 7	Lines 1 and 2
17+8 r==13+6 r	9+2 j1+2 j2+2 r==8+j2+3 r
21+10 r==1+2 (12+4 r)	12+j1+2 j2+4 r==1+2 (9+4 r)
No solutions	No solutions
-----	-----
Lines 11 and 8	Lines 1 and 3
21+8 r==16+6 r	7+2 j1+2 j2==14+j2+6 r
25+10 r==1+2 (12+4 r)	10+j1+2 j2+2 r==1+2 (9+4 r)
No solutions	No solutions
-----	-----
Lines 11 and 9	Lines 1 and 4
18+8 r==19+8 r	8+2 j1+2 j2+2 r==15+j2+7 r
22+10 r==1+2 (12+4 r)	11+j1+2 j2+4 r==1+2 (9+4 r)
No solutions	No solutions
-----	-----
Lines 11 and 10	Lines 1 and 5
True	10+2 j1+4 r==14+7 r
19+8 r==1+2 (12+4 r)	13+j1+6 r==1+2 (9+4 r)

No solutions

-----

Lines 1 and 6

$12+2 \ j_1+4 \ r==17+8 \ r$

$15+j_1+6 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 1 and 7

$11+2 \ j_1+4 \ r==12+6 \ r$

$14+j_1+6 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 1 and 8

$9+2 \ j_1+4 \ r==11+6 \ r$

$12+j_1+6 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 1 and 9

$7+2 \ j_1+2 \ r==13+6 \ r$

$10+j_1+4 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 1

$9+2 \ j_1+2 \ j_2+2 \ r==6+j_2+2 \ r$

$11+j_1+2 \ j_2+3 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 2

$12+2 \ j_1+2 \ j_2+4 \ r==8+j_2+3 \ r$

$14+j_1+2 \ j_2+5 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 3

$10+2 \ j_1+2 \ j_2+2 \ r==14+j_2+6 \ r$

$12+j_1+2 \ j_2+3 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 4

$11+2 \ j_1+2 \ j_2+4 \ r==15+j_2+7 \ r$

$13+j_1+2 \ j_2+5 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 5

$13+2 \ j_1+6 \ r==14+7 \ r$

$15+j_1+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 6

$15+2 \ j_1+6 \ r==17+8 \ r$

$17+j_1+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 7

$14+2 \ j_1+6 \ r==12+6 \ r$

$16+j_1+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 8

$12+2 \ j_1+6 \ r==11+6 \ r$

$14+j_1+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 2 and 9

$10+2 \ j_1+4 \ r==13+6 \ r$

$$12+j1+5 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 1

$$7+2 \text{ } j1+2 \text{ } j2==6+j2+2 \text{ } r$$

$$17+j1+2 \text{ } j2+6 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

```
{j1->ConditionalExpression[2 C[1],
  C[1] \[Element] Integers && C[1] >= 1],
 j2->ConditionalExpression[3+2 C[1],
  C[1] \[Element] Integers && C[1] >= 1],
 r->ConditionalExpression[2+3 C[1],
  C[1] \[Element] Integers && C[1] >= 1]}}
```

-----

Lines 3 and 2

$$10+2 \text{ } j1+2 \text{ } j2+2 \text{ } r==8+j2+3 \text{ } r$$

$$20+j1+2 \text{ } j2+8 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 3

$$8+2 \text{ } j1+2 \text{ } j2==14+j2+6 \text{ } r$$

$$18+j1+2 \text{ } j2+6 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 4

$$9+2 \text{ } j1+2 \text{ } j2+2 \text{ } r==15+j2+7 \text{ } r$$

$$19+j1+2 \text{ } j2+8 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 5

$$11+2 \text{ } j1+4 \text{ } r==14+7 \text{ } r$$

$$21+j1+10 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 6

$$13+2 \text{ } j1+4 \text{ } r==17+8 \text{ } r$$

$$23+j1+10 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 7

$$12+2 \text{ } j1+4 \text{ } r==12+6 \text{ } r$$

$$22+j1+10 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 8

$$10+2 \text{ } j1+4 \text{ } r==11+6 \text{ } r$$

$$20+j1+10 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 3 and 9

$$8+2 \text{ } j1+2 \text{ } r==13+6 \text{ } r$$

$$18+j1+8 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 4 and 1

$$8+2 \text{ } j1+2 \text{ } j2+2 \text{ } r==6+j2+2 \text{ } r$$

$$18+j1+2 \text{ } j2+7 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 4 and 2

$$11+2 \text{ } j1+2 \text{ } j2+4 \text{ } r==8+j2+3 \text{ } r$$

$$21+j1+2 \text{ } j2+9 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 4 and 3

$$9+2 \text{ } j1+2 \text{ } j2+2 \text{ } r==14+j2+6 \text{ } r$$

$$19+j1+2 \text{ } j2+7 \text{ } r==1+2 \text{ } (9+4 \text{ } r)$$

No solutions

-----

Lines 4 and 4

$10+2 \ j_1+2 \ j_2+4 \ r==15+j_2+7 \ r$

$20+j_1+2 \ j_2+9 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 4 and 5

$12+2 \ j_1+6 \ r==14+7 \ r$

$22+j_1+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 4 and 6

$14+2 \ j_1+6 \ r==17+8 \ r$

$24+j_1+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 4 and 7

$13+2 \ j_1+6 \ r==12+6 \ r$

$23+j_1+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 4 and 8

$11+2 \ j_1+6 \ r==11+6 \ r$

$21+j_1+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 4 and 9

$9+2 \ j_1+4 \ r==13+6 \ r$

$19+j_1+9 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 1

$10+2 \ j_2+4 \ r==6+j_2+2 \ r$

$17+2 \ j_2+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 2

$13+2 \ j_2+6 \ r==8+j_2+3 \ r$

$20+2 \ j_2+9 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 3

$11+2 \ j_2+4 \ r==14+j_2+6 \ r$

$18+2 \ j_2+7 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 4

$12+2 \ j_2+6 \ r==15+j_2+7 \ r$

$19+2 \ j_2+9 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 5

$14+8 \ r==14+7 \ r$

$21+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 6

$16+8 \ r==17+8 \ r$

$23+11 \ r==1+2 \ (9+4 \ r)$

No solutions

-----

Lines 5 and 7

$15+8 \ r==12+6 \ r$

22+11  $r=1+2$  (9+4  $r$ )

No solutions

Lines 5 and 8

13+8  $r=11+6$   $r$

20+11  $r=1+2$  (9+4  $r$ )

No solutions

Lines 5 and 9

11+6  $r=13+6$   $r$

18+9  $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 1

12+2  $j^2+4$   $r=6+j^2+2$   $r$

20+2  $j^2+8$   $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 2

15+2  $j^2+6$   $r=8+j^2+3$   $r$

23+2  $j^2+10$   $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 3

13+2  $j^2+4$   $r=14+j^2+6$   $r$

21+2  $j^2+8$   $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 4

14+2  $j^2+6$   $r=15+j^2+7$   $r$

22+2  $j^2+10$   $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 5

16+8  $r=14+7$   $r$

24+12  $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 6

18+8  $r=17+8$   $r$

26+12  $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 7

17+8  $r=12+6$   $r$

25+12  $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 8

15+8  $r=11+6$   $r$

23+12  $r=1+2$  (9+4  $r$ )

No solutions

Lines 6 and 9

True

21+10  $r=1+2$  (9+4  $r$ )

No solutions

Lines 7 and 1

11+2  $j^2+4$   $r=6+j^2+2$   $r$

15+2  $j^2+6$   $r=1+2$  (9+4  $r$ )

No solutions

Lines 7 and 2



14+2 j2+6 r==8+j2+3 r	No solutions
18+2 j2+8 r==1+2 (9+4 r)	-----
No solutions	Lines 7 and 9
-----	12+6 r==13+6 r
Lines 7 and 3	16+8 r==1+2 (9+4 r)
12+2 j2+4 r==14+j2+6 r	No solutions
16+2 j2+6 r==1+2 (9+4 r)	-----
No solutions	Lines 8 and 1
-----	9+2 j2+4 r==6+j2+2 r
Lines 7 and 4	14+2 j2+6 r==1+2 (9+4 r)
13+2 j2+6 r==15+j2+7 r	No solutions
17+2 j2+8 r==1+2 (9+4 r)	-----
No solutions	Lines 8 and 2
-----	12+2 j2+6 r==8+j2+3 r
Lines 7 and 5	17+2 j2+8 r==1+2 (9+4 r)
15+8 r==14+7 r	No solutions
19+10 r==1+2 (9+4 r)	-----
No solutions	Lines 8 and 3
-----	10+2 j2+4 r==14+j2+6 r
Lines 7 and 6	15+2 j2+6 r==1+2 (9+4 r)
True	No solutions
21+10 r==1+2 (9+4 r)	-----
No solutions	Lines 8 and 4
-----	11+2 j2+6 r==15+j2+7 r
Lines 7 and 7	16+2 j2+8 r==1+2 (9+4 r)
16+8 r==12+6 r	No solutions
20+10 r==1+2 (9+4 r)	-----
No solutions	Lines 8 and 5
-----	13+8 r==14+7 r
Lines 7 and 8	18+10 r==1+2 (9+4 r)
14+8 r==11+6 r	No solutions
18+10 r==1+2 (9+4 r)	-----

Lines 8 and 6

$$15+8 \ r==17+8 \ r$$

$$20+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 8 and 7

$$14+8 \ r==12+6 \ r$$

$$19+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 8 and 8

$$12+8 \ r==11+6 \ r$$

$$17+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 8 and 9

$$10+6 \ r==13+6 \ r$$

$$15+8 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 1

$$7+2 \ j2+2 \ r==6+j2+2 \ r$$

$$16+2 \ j2+6 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 2

$$10+2 \ j2+4 \ r==8+j2+3 \ r$$

$$19+2 \ j2+8 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 3

$$8+2 \ j2+2 \ r==14+j2+6 \ r$$

$$17+2 \ j2+6 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 4

$$9+2 \ j2+4 \ r==15+j2+7 \ r$$

$$18+2 \ j2+8 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 5

$$11+6 \ r==14+7 \ r$$

$$20+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 6

$$13+6 \ r==17+8 \ r$$

$$22+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 7

True

$$21+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 8

$$10+6 \ r==11+6 \ r$$

$$19+10 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----

Lines 9 and 9

$$8+4 \ r==13+6 \ r$$

$$17+8 \ r==1+2 \ (9+4 \ r)$$

No solutions

-----  
 \*\*\*\*\*  
 Theorem 4.4 Case  $n = 4k+2$   
 Lines 1 and 1  
 $8+2 j_1+2 j_2==5+j_2+2 r$   
 $9+j_1+2 j_2+2 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 2  
 $5+2 j_1+2 j_2+2 r==4+j_2+3 r$   
 $6+j_1+2 j_2+4 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 3  
 $7+2 j_1+2 j_2==5+j_2+6 r$   
 $8+j_1+2 j_2+2 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 4  
 $6+2 j_1+2 j_2+2 r==5+j_2+7 r$   
 $7+j_1+2 j_2+4 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 5  
 $4+2 j_1+2 r==4+4 r$   
 $5+j_1+4 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 6  
 $5+2 j_1+4 r==4+7 r$   
 $6+j_1+6 r==1+2 (2+4 r)$   
 No solutions

-----  
 Lines 1 and 7  
 $6+2 j_1+4 r==4+6 r$   
 $7+j_1+6 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 1 and 8  
 $4+2 j_1+4 r==3+6 r$   
 $5+j_1+6 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 2 and 1  
 $5+2 j_1+2 j_2+2 r==5+j_2+2 r$   
 $8+j_1+2 j_2+3 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 2 and 2  
 $2+2 j_1+2 j_2+4 r==4+j_2+3 r$   
 $5+j_1+2 j_2+5 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 2 and 3  
 $4+2 j_1+2 j_2+2 r==5+j_2+6 r$   
 $7+j_1+2 j_2+3 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 2 and 4  
 $3+2 j_1+2 j_2+4 r==5+j_2+7 r$   
 $6+j_1+2 j_2+5 r==1+2 (2+4 r)$   
 No solutions  
 -----  
 Lines 2 and 5

$$1+2 \ j1+4 \ r==4+4 \ r$$

$$4+j1+5 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 2 and 6

$$2+2 \ j1+6 \ r==4+7 \ r$$

$$5+j1+7 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 2 and 7

$$3+2 \ j1+6 \ r==4+6 \ r$$

$$6+j1+7 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 2 and 8

$$1+2 \ j1+6 \ r==3+6 \ r$$

$$4+j1+7 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 1

$$7+2 \ j1+2 \ j2==5+j2+2 \ r$$

$$9+j1+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$$

```
{j1->ConditionalExpression[2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
 j2->ConditionalExpression[-2+2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
 r->ConditionalExpression[3 C[1],
  C[1] \[Element] Integers && C[1] >= 2]}}
```

-----

Lines 3 and 2

$$4+2 \ j1+2 \ j2+2 \ r==4+j2+3 \ r$$

$$6+j1+2 \ j2+8 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 3

$$6+2 \ j1+2 \ j2==5+j2+6 \ r$$

$$8+j1+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 4

$$5+2 \ j1+2 \ j2+2 \ r==5+j2+7 \ r$$

$$7+j1+2 \ j2+8 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 5

$$3+2 \ j1+2 \ r==4+4 \ r$$

$$5+j1+8 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 6

$$4+2 \ j1+4 \ r==4+7 \ r$$

$$6+j1+10 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 7

$$5+2 \ j1+4 \ r==4+6 \ r$$

$$7+j1+10 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 3 and 8

$$3+2 \ j1+4 \ r==3+6 \ r$$

$$5+j1+10 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 4 and 1

$$6+2 \ j1+2 \ j2+2 \ r==5+j2+2 \ r$$

$$9+j1+2 \ j2+7 \ r==1+2 \ (2+4 \ r)$$

No solutions

-----

Lines 4 and 2

$3+2 \ j1+2 \ j2+4 \ r==4+j2+3 \ r$

$6+j1+2 \ j2+9 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 3

$5+2 \ j1+2 \ j2+2 \ r==5+j2+6 \ r$

$8+j1+2 \ j2+7 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 4

$4+2 \ j1+2 \ j2+4 \ r==5+j2+7 \ r$

$7+j1+2 \ j2+9 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 5

$2+2 \ j1+4 \ r==4+4 \ r$

$5+j1+9 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 6

$3+2 \ j1+6 \ r==4+7 \ r$

$6+j1+11 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 7

$4+2 \ j1+6 \ r==4+6 \ r$

$7+j1+11 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 4 and 8

$2+2 \ j1+6 \ r==3+6 \ r$

$5+j1+11 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 1

$4+2 \ j2+2 \ r==5+j2+2 \ r$

$8+2 \ j2+4 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 2

$1+2 \ j2+4 \ r==4+j2+3 \ r$

$5+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 3

$3+2 \ j2+2 \ r==5+j2+6 \ r$

$7+2 \ j2+4 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 4

$2+2 \ j2+4 \ r==5+j2+7 \ r$

$6+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 5

$4 \ r==4+4 \ r$

$4+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 5 and 6

$1+6 \ r==4+7 \ r$

5+8  $r=1+2(2+4r)$

No solutions

Lines 5 and 7

2+6  $r=4+6r$

6+8  $r=1+2(2+4r)$

No solutions

Lines 5 and 8

6  $r=3+6r$

4+8  $r=1+2(2+4r)$

No solutions

Lines 6 and 1

5+2  $j_2+4r=5+j_2+2r$

8+2  $j_2+7r=1+2(2+4r)$

No solutions

Lines 6 and 2

2+2  $j_2+6r=4+j_2+3r$

5+2  $j_2+9r=1+2(2+4r)$

No solutions

Lines 6 and 3

4+2  $j_2+4r=5+j_2+6r$

7+2  $j_2+7r=1+2(2+4r)$

No solutions

Lines 6 and 4

3+2  $j_2+6r=5+j_2+7r$

6+2  $j_2+9r=1+2(2+4r)$

No solutions

Lines 6 and 5

1+6  $r=4+4r$

4+9  $r=1+2(2+4r)$

No solutions

Lines 6 and 6

2+8  $r=4+7r$

5+11  $r=1+2(2+4r)$

No solutions

Lines 6 and 7

3+8  $r=4+6r$

6+11  $r=1+2(2+4r)$

No solutions

Lines 6 and 8

1+8  $r=3+6r$

4+11  $r=1+2(2+4r)$

No solutions

Lines 7 and 1

6+2  $j_2+4r=5+j_2+2r$

8+2  $j_2+6r=1+2(2+4r)$

No solutions

Lines 7 and 2

3+2  $j_2+6r=4+j_2+3r$

5+2  $j_2+8r=1+2(2+4r)$

No solutions

Lines 7 and 3

$5+2 \ j2+4 \ r==5+j2+6 \ r$

$7+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 7 and 4

$4+2 \ j2+6 \ r==5+j2+7 \ r$

$6+2 \ j2+8 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 7 and 5

$2+6 \ r==4+4 \ r$

$4+8 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 7 and 6

$3+8 \ r==4+7 \ r$

$5+10 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 7 and 7

$4+8 \ r==4+6 \ r$

$6+10 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 7 and 8

$2+8 \ r==3+6 \ r$

$4+10 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 1

$4+2 \ j2+4 \ r==5+j2+2 \ r$

$7+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 2

$1+2 \ j2+6 \ r==4+j2+3 \ r$

$4+2 \ j2+8 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 3

$3+2 \ j2+4 \ r==5+j2+6 \ r$

$6+2 \ j2+6 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 4

$2+2 \ j2+6 \ r==5+j2+7 \ r$

$5+2 \ j2+8 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 5

$6 \ r==4+4 \ r$

$3+8 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 6

$1+8 \ r==4+7 \ r$

$4+10 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 7

$2+8 \ r==4+6 \ r$

$5+10 \ r==1+2 \ (2+4 \ r)$

No solutions

-----

Lines 8 and 8

$$8 \ r == 3+6 \ r$$

$$3+10 \ r == 1+2 \ (2+4 \ r)$$

No solutions

-----

\*\*\*\*\*

Theorem 4.4 Case  $n = 4k+3$

Lines 1 and 1

$$8+2 \ j_1+2 \ j_2 == 7+j_2+2 \ r$$

$$11+j_1+2 \ j_2+2 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 2

$$7+2 \ j_1+2 \ j_2+2 \ r == 6+j_2+3 \ r$$

$$10+j_1+2 \ j_2+4 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 3

$$5+2 \ j_1+4 \ r == 5+4 \ r$$

$$8+j_1+6 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 4

$$5+2 \ j_1+2 \ r == 6+4 \ r$$

$$8+j_1+4 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 5

$$6+2 \ j_1+4 \ r == 6+5 \ r$$

$$9+j_1+6 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 6

$$4+2 \ j_1+4 \ r == 6+6 \ r$$

$$7+j_1+6 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 7

$$7+2 \ j_1+4 \ r == 7+6 \ r$$

$$10+j_1+6 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 8

$$7+2 \ j_1+2 \ j_2 == 8+j_2+6 \ r$$

$$10+j_1+2 \ j_2+2 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 1 and 9

$$6+2 \ j_1+2 \ j_2+2 \ r == 7+j_2+7 \ r$$

$$9+j_1+2 \ j_2+4 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 2 and 1

$$7+2 \ j_1+2 \ j_2+2 \ r == 7+j_2+2 \ r$$

$$10+j_1+2 \ j_2+3 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 2 and 2

$$6+2 \ j_1+2 \ j_2+4 \ r == 6+j_2+3 \ r$$

$$9+j_1+2 \ j_2+5 \ r == 1+2 \ (3+4 \ r)$$

No solutions

-----

Lines 2 and 3

$$4+2 \ j_1+6 \ r == 5+4 \ r$$



$$7+j_1+7 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 4

$$4+2 \ j_1+4 \ r==6+4 \ r$$

$$7+j_1+5 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 5

$$5+2 \ j_1+6 \ r==6+5 \ r$$

$$8+j_1+7 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 6

$$3+2 \ j_1+6 \ r==6+6 \ r$$

$$6+j_1+7 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 7

$$6+2 \ j_1+6 \ r==7+6 \ r$$

$$9+j_1+7 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 8

$$6+2 \ j_1+2 \ j_2+2 \ r==8+j_2+6 \ r$$

$$9+j_1+2 \ j_2+3 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 2 and 9

$$5+2 \ j_1+2 \ j_2+4 \ r==7+j_2+7 \ r$$

$$8+j_1+2 \ j_2+5 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 1

$$5+2 \ j_2+4 \ r==7+j_2+2 \ r$$

$$9+2 \ j_2+4 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 2

$$4+2 \ j_2+6 \ r==6+j_2+3 \ r$$

$$8+2 \ j_2+6 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 3

$$2+8 \ r==5+4 \ r$$

$$6+8 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 4

$$2+6 \ r==6+4 \ r$$

$$6+6 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 5

$$3+8 \ r==6+5 \ r$$

$$7+8 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 6

$$1+8 \ r==6+6 \ r$$

$$5+8 \ r==1+2 \ (3+4 \ r)$$

No solutions

Lines 3 and 7

4+8 r==7+6 r

8+8 r==1+2 (3+4 r)

No solutions

-----

Lines 3 and 8

4+2 j2+4 r==8+j2+6 r

8+2 j2+4 r==1+2 (3+4 r)

No solutions

-----

Lines 3 and 9

3+2 j2+6 r==7+j2+7 r

7+2 j2+6 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 1

5+2 j2+2 r==7+j2+2 r

10+2 j2+4 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 2

4+2 j2+4 r==6+j2+3 r

9+2 j2+6 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 3

2+6 r==5+4 r

7+8 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 4

2+4 r==6+4 r

7+6 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 5

3+6 r==6+5 r

8+8 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 6

1+6 r==6+6 r

6+8 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 7

4+6 r==7+6 r

9+8 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 8

4+2 j2+2 r==8+j2+6 r

9+2 j2+4 r==1+2 (3+4 r)

No solutions

-----

Lines 4 and 9

3+2 j2+4 r==7+j2+7 r

8+2 j2+6 r==1+2 (3+4 r)

No solutions

-----

Lines 5 and 1

6+2 j2+4 r==7+j2+2 r

10+2 j2+5 r==1+2 (3+4 r)

No solutions

-----

Lines 5 and 2

$$5+2 \text{ } j2+6 \text{ } r==6+j2+3 \text{ } r$$

$$9+2 \text{ } j2+7 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 3

$$3+8 \text{ } r==5+4 \text{ } r$$

$$7+9 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 4

$$3+6 \text{ } r==6+4 \text{ } r$$

$$7+7 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 5

$$4+8 \text{ } r==6+5 \text{ } r$$

$$8+9 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 6

$$2+8 \text{ } r==6+6 \text{ } r$$

$$6+9 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 7

$$5+8 \text{ } r==7+6 \text{ } r$$

$$9+9 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 8

$$5+2 \text{ } j2+4 \text{ } r==8+j2+6 \text{ } r$$

$$9+2 \text{ } j2+5 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 5 and 9

$$4+2 \text{ } j2+6 \text{ } r==7+j2+7 \text{ } r$$

$$8+2 \text{ } j2+7 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 6 and 1

$$4+2 \text{ } j2+4 \text{ } r==7+j2+2 \text{ } r$$

$$10+2 \text{ } j2+6 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 6 and 2

$$3+2 \text{ } j2+6 \text{ } r==6+j2+3 \text{ } r$$

$$9+2 \text{ } j2+8 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 6 and 3

$$1+8 \text{ } r==5+4 \text{ } r$$

$$7+10 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 6 and 4

$$1+6 \text{ } r==6+4 \text{ } r$$

$$7+8 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----

Lines 6 and 5

$$2+8 \text{ } r==6+5 \text{ } r$$

$$8+10 \text{ } r==1+2 \text{ } (3+4 \text{ } r)$$

No solutions

-----  
 Lines 6 and 6  
 $8r = 6 + 6r$   
 $6 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 6 and 7  
 $3 + 8r = 7 + 6r$   
 $9 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 6 and 8  
 $3 + 2j_2 + 4r = 8 + j_2 + 6r$   
 $9 + 2j_2 + 6r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 6 and 9  
 $2 + 2j_2 + 6r = 7 + j_2 + 7r$   
 $8 + 2j_2 + 8r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 1  
 $7 + 2j_2 + 4r = 7 + j_2 + 2r$   
 $11 + 2j_2 + 6r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 2  
 $6 + 2j_2 + 6r = 6 + j_2 + 3r$   
 $10 + 2j_2 + 8r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 3

$4 + 8r = 5 + 4r$   
 $8 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 4  
 $4 + 6r = 6 + 4r$   
 $8 + 8r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 5  
 $5 + 8r = 6 + 5r$   
 $9 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 6  
 $3 + 8r = 6 + 6r$   
 $7 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 7  
 $6 + 8r = 7 + 6r$   
 $10 + 10r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 8  
 $6 + 2j_2 + 4r = 8 + j_2 + 6r$   
 $10 + 2j_2 + 6r = 1 + 2(3 + 4r)$   
 No solutions  
 -----  
 Lines 7 and 9  
 $5 + 2j_2 + 6r = 7 + j_2 + 7r$   
 $9 + 2j_2 + 8r = 1 + 2(3 + 4r)$

No solutions	8+j1+10 r==1+2 (3+4 r)
-----	No solutions
-----	-----
Lines 8 and 1	Lines 8 and 7
7+2 j1+2 j2==7+j2+2 r	6+2 j1+4 r==7+6 r
12+j1+2 j2+6 r==1+2 (3+4 r)	11+j1+10 r==1+2 (3+4 r)
{j1->ConditionalExpression[1+2 C[1], C[1] \[Element] Integers && C[1] >= 4], j2->ConditionalExpression[-4+2 C[1], C[1] \[Element] Integers && C[1] >= 4], r->ConditionalExpression[-1+3 C[1], C[1] \[Element] Integers && C[1] >= 4]}}	No solutions
-----	-----
Lines 8 and 2	Lines 8 and 8
6+2 j1+2 j2+2 r==6+j2+3 r	6+2 j1+2 j2==8+j2+6 r
11+j1+2 j2+8 r==1+2 (3+4 r)	11+j1+2 j2+6 r==1+2 (3+4 r)
No solutions	No solutions
-----	-----
Lines 8 and 3	Lines 8 and 9
4+2 j1+4 r==5+4 r	5+2 j1+2 j2+2 r==7+j2+7 r
9+j1+10 r==1+2 (3+4 r)	10+j1+2 j2+8 r==1+2 (3+4 r)
No solutions	No solutions
-----	-----
Lines 8 and 4	Lines 9 and 1
4+2 j1+2 r==6+4 r	6+2 j1+2 j2+2 r==7+j2+2 r
9+j1+8 r==1+2 (3+4 r)	11+j1+2 j2+7 r==1+2 (3+4 r)
No solutions	{j1->0,j2->1,r->6}}
-----	-----
Lines 8 and 5	Lines 9 and 2
5+2 j1+4 r==6+5 r	5+2 j1+2 j2+4 r==6+j2+3 r
10+j1+10 r==1+2 (3+4 r)	10+j1+2 j2+9 r==1+2 (3+4 r)
No solutions	No solutions
-----	-----
Lines 8 and 6	Lines 9 and 3
3+2 j1+4 r==6+6 r	3+2 j1+6 r==5+4 r
	8+j1+11 r==1+2 (3+4 r)
	No solutions

```

-----
Lines 9 and 4

3+2 j1+4 r==6+4 r

8+j1+9 r==1+2 (3+4 r)

No solutions

-----

Lines 9 and 5

4+2 j1+6 r==6+5 r

9+j1+11 r==1+2 (3+4 r)

No solutions

-----

Lines 9 and 6

2+2 j1+6 r==6+6 r

7+j1+11 r==1+2 (3+4 r)

No solutions

-----

Lines 9 and 7

5+2 j1+6 r==7+6 r

10+j1+11 r==1+2 (3+4 r)

No solutions

-----

Lines 9 and 8

5+2 j1+2 j2+2 r==8+j2+6 r

10+j1+2 j2+7 r==1+2 (3+4 r)

No solutions

-----

Lines 9 and 9

```

```

4+2 j1+2 j2+4 r==7+j2+7 r

9+j1+2 j2+9 r==1+2 (3+4 r)

No solutions

-----

*****

Exception Cases:

Theorem, First Line, Second Line, Solution

Theorem 4.2 Case n = 4k, 3, 1,
{{j1 -> ConditionalExpression[2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
  j2 -> ConditionalExpression[6 + 2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
  r -> ConditionalExpression[4 + 3 C[1],
  C[1] \[Element] Integers && C[1] >= 2]}}

Theorem 4.2 Case n = 4k, 3, 2,
{{j1 -> 1, j2 -> 0, r -> 5}}

Theorem 4.2 Case n = 4k+1, 3, 1,
{{j1 -> ConditionalExpression[2 C[1],
  C[1] \[Element] Integers && C[1] >= 1],
  j2 -> ConditionalExpression[3 + 2 C[1],
  C[1] \[Element] Integers && C[1] >= 1],
  r -> ConditionalExpression[2 + 3 C[1],
  C[1] \[Element] Integers && C[1] >= 1]}}

Theorem 4.4 Case n = 4k+2, 3, 1,
{{j1 -> ConditionalExpression[2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
  j2 -> ConditionalExpression[-2 + 2 C[1],
  C[1] \[Element] Integers && C[1] >= 2],
  r -> ConditionalExpression[3 C[1],
  C[1] \[Element] Integers && C[1] >= 2]}}

Theorem 4.4 Case n = 4k+3, 8, 1,
{{j1 -> ConditionalExpression[1 + 2 C[1],
  C[1] \[Element] Integers && C[1] >= 4],
  j2 -> ConditionalExpression[-4 + 2 C[1],
  C[1] \[Element] Integers && C[1] >= 4],
  r -> ConditionalExpression[-1 + 3 C[1],
  C[1] \[Element] Integers && C[1] >= 4]}}

Theorem 4.4 Case n = 4k+3, 9, 1,
{{j1 -> 0, j2 -> 1, r -> 6}}

```